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FOUNDATIONS

*Ackermann, W. **Solvable cases of the decision problem.** Studies in logic and the foundations of mathematics. North-Holland Publishing Co., Amsterdam, 1954. viii+114 pp.

The decision problem for a class C of (closed) formulae of the predicate calculus is here solved in the following sense: an algorithm is obtained which determines, for any formula of C, the cardinal numbers of the domains in which it is valid. The classes C considered are (1) the monadic predicate calculus with identity; prenex formulae with prefixes; (2) $(z_1)\cdots(z_p)(Ex_1)\cdots(Ex_n)$; (3) $(z_1)\cdots(z_p)(Ex)(y_1)\cdots(y_n)$; (4) $(z_1)\cdots(z_p)(Ex_1)(Ex_2)(y_1)\cdots(y_n)$, where the non-logical constants are predicate symbols and identity; (5) $(z_1)\cdots(z_p)$, where the non-logical constants are predicate and function symbols; (6) $(Ex_1)\cdots(Ex_n)$, where the non-logical constants are predicate symbols and a single monadic function symbol.

By a corollary to the completeness theorems of the predicate calculus [e.g. Henkin, J. Symbolic Logic **14**, 159–166 (1949); these Rev. **11**, 487], the problem is reduced as follows: if a formula is valid in a denumerable domain it is (a) universally valid unless it contains the identity symbol, (b) valid in all domains with cardinal exceeding some finite k; on the axiom of choice, if a formula is valid in some infinite domain it is also valid in every denumerable domain.

(1) is solved by the elimination of quantifiers. The solution provides a complete description of all sets definable in the predicate calculus whose constants are monadic predicates; it is easily seen that no such elimination procedure is possible for cases (2)–(6). By a remark due to Kálmár [Math. Ann. **108**, 466–484 (1933)] cases (2)–(4) are solved if there is a decision procedure for these prefixes without the variable z. The author treats these cases in two steps. First, the formulae are so modified that distinct variables refer to distinct elements, e.g., $(Ex_1)\cdots(Ex_n)B(x_1\cdots x_n)$ is replaced by $(Ex_1)\cdots(Ex_n)\mathfrak{B}'(x_1\cdots x_n)$, where $\mathfrak{B}'(a_1\cdots a_n)$ is the disjunction of all the formulae $B(a_1\cdots a_n)$, and the a's denote (not necessarily distinct) x's; this step permits an extension of Herbrand's theorem to formulae with the identity symbol: if $(x_1)\cdots(x_n)(Ey_1)\cdots(Ey_m)A(x_1\cdots x_n y_1\cdots y_m)$ can be satisfied at all, then also over the integers with $y_i=\varphi_i(x_1\cdots x_n)$ and φ_i disparate [Hilbert and Bernays, Bd II, Springer, Berlin, 1939, p. 168]. Second, related formulae of the monadic predicate calculus are constructed which can be decided by (1): the main difficulty here is the choice of suitable auxiliary constants.

In (2) a special case of a powerful combinatorial principle due to F. P. Ramsey [Proc. London Math. Soc. (2) **30**, 264–286 (1929)] explains the choice of auxiliary constants: for any given relations $\Gamma_j(a, b)$ defined in a sufficiently large domain D of the integers, there is a subdomain D_1 such that, for a, b in D_1, $\Gamma_j(a, b)$ depends only on a if $a<b$. In $(Ex_1)\cdots(Ex_n)B(x_1\cdots x_n)(\mathfrak{B})$, without loss of generality, x_i may be taken $<x_j$ for $i<j$; if in $\mathfrak{B}$ (dyadic) predicates $G_p(x_i, x_j)$ are replaced by $\Phi_p(x_i)$ if $i=j$, $\Psi_p(x_i)$ if $i<j$, $\Theta_p(x_i)$ if $j<i$ to yield the monadic formula $\mathfrak{B}_*$, $\mathfrak{B}_*$ is valid in D_1 if $\mathfrak{B}$ is valid in D, and $\mathfrak{B}$ is valid in D if $\mathfrak{B}_*$ is valid in D (Ramsey's paper is not cited).

In (3) it is best to consider the dual

$$(x)(Ey_1)\cdots(Ey_n)A(xy_1\cdots y_n)(\mathfrak{A}).$$

If (for simplicity) the non-logical constants of $\mathfrak{A}$ are dyadic $G_i(a, b)$, $i\leqq k$, $G_i(a, a)$ is replaced by $H_i(a)$; $G_i(x, y_p)$, $G_i(y_p, x)$, $G_i(y_p, y_q)$ $(p\neq q)$ by $K_{ip}(x)$, $L_{ip}(x)$, $M_{ipq}(x)$ to yield the monadic formula

$$(x)(Ey_1)\cdots(Ey_n)A_1(xy_1\cdots y_n)(\mathfrak{A}_1).$$

If this can be satisfied at all, then also by disparate functions $y_i=\phi_i(x)$ over the integers; the definitions

$$G_i^*(a, a)\leftrightarrow H_i(a),\quad G_i^*[x, \phi_p(x)]\leftrightarrow K_{ip}(x),$$
$$G_i^*[\phi_p(x), x]\leftrightarrow L_{ip}(x),\quad G^*[\phi_p(x), \phi_q(x)]\leftrightarrow M_{ipq}(x)$$

define G_i^* consistently since, by disparateness of ϕ, the arguments of G_i^* do not clash, and $A^*[x, \phi_1(x)\cdots\phi_n(x)]$ holds. Analysis of this step from $\mathfrak{A}_1$ back to $\mathfrak{A}$ shows that it is not necessary that ϕ be disparate, but only that $\phi_p[\phi_q(x)]\neq x$ (which is a consequence of disparateness): for this it is sufficient that the elements of the domain of $\mathfrak{A}_1$ can be divided into exclusive and exhaustive classes N_0, N_1, N_2 where $\phi_p(x)$ (i.e. y_p) $\varepsilon\, N_i$ if $x\ \varepsilon\ N_{(i-1) \bmod 3}$. By adding the appropriate (monadic) axioms concerning N to $\mathfrak{A}_1$ to yield $\mathfrak{A}_2$, one can step from a model of $\mathfrak{A}_2$ in a finite domain to a model of $\mathfrak{A}$ in the same domain while there is no disparate set of functions whose values all lie in the finite domain in which they are defined.

The basic idea of the solution for (4) is quite simple, but the treatment complicated: for $p=0$, $n=2$, a dyadic predicate as single non-logical constant, over 50 auxiliary symbols and over 1000 auxiliary axioms are needed. (5) is in effect a quantifier-free formula, and therefore valid only if provable by means of substitution in an identical formula of the propositional calculus and a specifiable set of identity formulae, which is decidable [see, e.g., Hilbert and Bernays, loc. cit., p. 80, first ϵ-theorem which applies to the predicate calculus with both function and predicate symbols]. The solution of (6) is an application of the ideas used in the proof of Herbrand's theorem.

In the reviewer's opinion the formal introduction of monadic predicates in cases (2)–(4) is quite artificial. The novel ideas introduced by the author seem more at home in the following type of treatment, due to Herbrand and others [cf. Church, Rev. Philos. Louvain **49**, 203–221 (1951)]. In case (3): if $\mathfrak{A}$ can be satisfied, then also by disparate functions ψ_i over the integers; let $0\ \varepsilon\ N_0$ and $\psi_p(x)\ \varepsilon\ N_{(i+1) \bmod 3}$ if $x\ \varepsilon\ N_i$ (this determines the N-class of each integer uniquely since, by disparateness, the expression of x in terms of ψ and 0 is unique); let x^* be the least equivalent of x, i.e., $x\ \varepsilon\ N_i\leftrightarrow x^*\ \varepsilon\ N_i$, and $G_j(x, x)\leftrightarrow G_j(x^*, x^*)$ for $j\leqq k$; define

$$\psi^*(x)=\psi(x^*),\quad G^*(x, x)\leftrightarrow G(x, x),$$
$$G^*[x, \psi^*(x)]\leftrightarrow G[x^*, \psi(x^*)]\quad \text{etc.}$$

There are at most 2^k values of x^* in each N-class, and $\leqq 3n2^k$ values of $\psi^*(x)$. $\mathfrak{A}$ is satisfied by G^* and ψ^* in the domain of integers which are values of $\psi^*(x)$, i.e. in a domain with $\leqq 3n2^k$ elements. The decision procedure of (1) for $\mathfrak{A}_2$ requires a domain of $2^{3+k[1+3n+\frac{1}{2}n(n+1)]}$ elements. The case (4) requires a subdivision of the integers into 7 classes (k_i on p. 78) and the introduction of "least equivalent pairs" (p. 77). Each formula of (1)–(5) is such that it is valid in a denumerable domain if it is valid in a sufficiently large finite domain. This is not true of many quantification theories of mathematical concepts which are decidable by the method of elimination of quantifiers, but have no finite model, e.g. the theory of densely ordered sets.

G. Kreisel (Reading).

Kalicki, Jan. An undecidable problem in the algebra of truth-tables. J. Symbolic Logic **19**, 172–176 (1954).

Two truth-tables are defined to be "equal" if they determine the same set of tautologies. Kalicki first shows that the problem of deciding whether two infinite-valued truth-tables are equal is unsolvable. He then defines a truth-table to be recursive if, under a suitable arithmetization, the set of tautologies of the table is recursive and shows that if the problem is restricted to recursive tables it is still unsolvable.

A. Rose (Nottingham).

✱Bereczki, Ilona. Existenz einer nichtelementaren rekursiven Funktion. Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, pp. 409–417. Akadémiai Kiadó, Budapest, 1952. (Hungarian and German. Russian summary)

According to a concept due to L. Kálmár, a function on the non-negative integers is called elementary, if it can be obtained in a finite number of steps by starting with the integer 1 and the following functions:

$$F_1(a, b)=a+b, \quad F_2(a, b)=|a-b|,$$
$$F_3(a, b)=a\cdot b, \quad F_4(a, b)=[a/b],$$

and the two schemes on formations of sums and products:

$$F_6(n, a, b, \cdots)=\sum_{i=1}^{n}\alpha(i, a, b, \cdots),$$

$$F_7(n, a, b, \cdots)=\prod_{i=1}^{n}\alpha(i, a, b, \cdots),$$

where α is a previously defined elementary function. Every elementary function is primitive recursive. It is shown here that the converse does not hold: there are primitive recursive functions which are not elementary. In fact, the primitive recursive function defined by the recursion $f(a, 0)=1, f(a, n+1)=a^{f(a,n)}$ is shown to be non-elementary by an argument based on the rate of growth of this function.

I. Novak Gál (Ithaca, N. Y.).

Skolem, Th. Remarks on "elementary" arithmetic functions. Norske Vid. Selsk. Forh., Trondheim **27**, no. 6, 6 pp. (1954).

It is shown here that the set of elementary functions E of L. Kálmár can be defined by taking as starting functions $F_1(x, y)=x+y$, $F_2(x, y)=x\cdot y$, $F_3(x, y)=\delta(x, y)$, where $\delta(x, y)=0$ if $x\neq y$, $\delta(x, y)=1$ if $x=y$, and the sum and product scheme are both allowed. The elementary functions can also be generated starting from 0 and 1 and the functions $F_1(x, y)=x+y$, $F_2(x, y)=x^y$, $F_3(x, y)=\delta(x, y)$ and the summation scheme, without using the product scheme at all. All the functions which occur in arithmetic are elementary functions, but when we get to analysis we may get beyond the reach of elementary functions.

I. Novak Gál.

Quine, W. V. Quantification and the empty domain. J. Symbolic Logic **19**, 177–179 (1954).

Will man in der Quantorenlogik (=elementarer Prädikatenlogik) genau die Formeln als Sätze haben, die in jedem Individuenbereich, einschliesslich des leeren, allgemeingültig sind, so hat man für eine falsche Formel f, in der x nicht vorkommt, im leeren Bereich für $(x)f$ die Wahl zwischen "wahr" und "falsch". Verf. empfiehlt "wahr", damit $(x)f\equiv(x)(f\wedge(Fx\supset Fx))$ gültig bleibt. Als Kalkül genügt für diese Wahl der Kalkül des Verf. aus "Mathematical logic" [rev. ed., Harvard, 1951; diese Rev. **13**, 613], wenn "$(\alpha)\phi\supset\phi\frac{\beta}{\alpha}$" ersetzt wird durch "$(\alpha)\phi\supset\phi\frac{\beta}{\alpha}$, wenn α frei in ϕ vorkommt."

P. Lorenzen (Bonn).

Beneš, Václav Edvard. A partial model for Quine's "New foundations". J. Symbolic Logic **19**, 197–200 (1954).

The author constructs a denumerable model for part of Quine's New Foundations NF [Amer. Math. Monthly **44**, 70–80 (1937)]. Hailperin [J. Symbolic Logic **9**, 1–19 (1944); these Rev. **5**, 197] has given a set of 9 axioms P1, $\cdots$, P9 characterizing NF. The model given here satisfies the first 8 of these but fails to satisfy P9: $(E\beta)(x, y)[\langle\iota x, y\rangle\,\epsilon\,\beta\equiv x\,\epsilon\,y]$. The construction proceeds by taking the set ω of all natural numbers, letting 0 represent the "universe set", 1 the "cardinal 1", and then passing to the closure under the operations implicit in P1 to P8. The construction parallels that of Gödel [The consistency of the continuum hypothesis, Princeton, 1940; these Rev. **2**, 66].

I. Novak Gál.

Aubert, Karl Egil. On the foundation of the theory of relations and the logical independence of generalized concepts of reflexivity, symmetry and transitivity. Arch. Math. Naturvid. **52**, 9–56 (1954).

R is a mapping of the set M^n of the n-tuples of M into a set L_κ with cardinal $\kappa\geqq 2$. Let R_i be the subset of M^n mapped into the element T_i of L_κ. The mapping R is (i) i-reflexive if all tuples consisting of identical elements of M are in R_i, (ii) (S, i)-symmetric, for a set S of permutations, if R_i is S-symmetric, (iii) i-transitive if $n=2m$ and $\langle a_1\cdots a_m b_1\cdots b_m\rangle$ is in R_i if, for some $c_1, \cdots, c_m$, both $\langle a_1\cdots a_m c_1\cdots c_m\rangle$ and $\langle c_1\cdots c_m b_1\cdots b_m\rangle$ are in R_i. Properties P, Q, R are logically independent if each of the 8 conjunctions $(-1)^{i_1}P\&(-1)^{i_2}Q\&(-1)^{i_3}R$, with $i=0, 1$, is consistent.

Using these definitions, the author shows that for $\kappa\geqq 2$ and any set M with >2 elements, reflexivity, symmetry, and transitivity are logically independent concepts. If L_κ is the set of truth values of a κ-valued logic, natural variants of the above definitions of these concepts are introduced, and it is shown that, for $n=2$, they are logically independent if M contains $\geqq\kappa$ elements. (The above definitions reduce to the ordinary notion of relation if $\kappa=2$, i.e. a two-valued logic: the author's relations would ordinarily be regarded as a class of relations depending on a parameter T_i.)

G. Kreisel (Reading).

Davis, Chandler. Modal operators, equivalence relations, and projective algebras. Amer. J. Math. **76**, 747–762 (1954).

Davis introduces a modal operator C on Boolean algebras. An $S4$ operator is defined to be one satisfying $Ca\supseteq a$, $C0=0$, $CCa=Ca$, $C(a\cup b)=Ca\cup Cb$. An $S5$ operator is one

which, in addition, satisfies "$a \cap Cb = 0$ implies $Ca \cap Cb = 0$." After developing some properties of $S5$ operators he discusses the reason for considering Boolean algebras with several $S5$ operators. He then constructs two $S5$ operators in terms of a fixed element e of a projective algebra $\mathfrak{L}$ and shows that every Boolean algebra with two $S5$ operators is obtained in this way. Finally, he gives the relation of Boolean algebras with several $S5$ operators to the first-order functional calculus of logic. *A. Rose* (Nottingham).

Maehara, Shôji. Eine Darstellung der intuitionistischen Logik in der klassischen. Nagoya Math. J. 7, 45–64 (1954).

The author extends Gentzen's calculi LK and LJ by adjoining to them a unary operator Bew and the rules

$$BES\frac{\Gamma' \to \mathfrak{A}}{\Gamma' \to \text{Bew}\ \mathfrak{A}}, \quad BEA\frac{\mathfrak{A}, \Gamma \to \Theta}{\text{Bew}\ \mathfrak{A}, \Gamma \to \Theta}.$$

Here Γ' is subject to the restriction that it contains only formulae in which Bew is the last operator. Let BLK, BLJ be the calculi obtained by this extension. From a formula $\mathfrak{A}$ the formula $\mathfrak{A}^b$ is obtained by placing "Bew" before every subformula of $\mathfrak{A}$. If Γ is a sequence of formulae, Γ^b means the sequence which is obtained by replacing every formula $\mathfrak{A}$ in Γ by $\mathfrak{A}^b$. The main result of the paper is: If "Bew" does not occur in Γ and in θ, then $\Gamma \to \theta$ is deducible in LJ if and only if $\Gamma^b \to \theta^b$ is deducible in BLK. The author does not mention the obviously necessary condition that θ consists of not more than one formula. *A. Heyting* (Amsterdam).

Gnedenko, B. V. On the struggle of materialism with idealism in mathematics. Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) 8, no. 3(10), 68–80 (1954). (Romanian)

Translated from Visnik Akad. Nauk Ukraïn. SSR 1953, no. 11.

ALGEBRA

Have

★Levi, Howard. Elements of algebra. Chelsea Publishing Company, New York, 1954. 160 pp. $3.25.

This book is the text for a one-semester three-point mathematics course at the School of General Studies of Columbia University. Its aim is, roughly, to give a rigorous account of the fundamental concepts of algebra, including the natural numbers (here called cardinals), the integers, rationals and reals. There is an extensive treatment of the basic laws of algebra, of expressions involving one or several variables and of polynomials. In this set-up equality of polynomials is first defined as equality for all admissible values of the variables, and it is later proved, at least for the rational field, that this implies equality of corresponding coefficients. The chapter on real numbers is much more difficult than the rest of the book (and might well have caused the heart attack in one of the author's students referred to in the Preface). In the latter part of the book a good deal has to remain unproved, but as a compensation the reader is given a glimpse of the land where mathematicians are still exploring.

Two minor criticisms are that in the treatment of simultaneous equations (p. 115) it should have been verified that the values for the unknowns obtained do in fact satisfy the equations, and in the proof of Theorem I (middle of p. 47) it is not made sufficiently clear that all cases have been considered.

The book is written in a lucid and lively style. It will be welcomed by those who have a genuine interest in mathematics and want to know more about the foundations of our subject. Each chapter is illustrated by examples, and a number of exercises enable the reader to consolidate his knowledge. *W. Ledermann* (Manchester).

Nanjundiah, T. S. On a formula of Grosswald. Amer. Math. Monthly 61, 700–702 (1954).

An elementary proof is given for the formula

$$\sum_{k=0}^{n-r} (-2)^{-k}\binom{n}{r+k}\binom{n+r+k}{k} = \begin{cases} (-\tfrac{1}{2})^m \binom{n}{m} & \text{if } r \equiv n \pmod 2 \\ 0 & \text{if } r \not\equiv n \pmod 2 \end{cases} \quad \text{where } m = \tfrac{1}{2}(n-r)$$

[for previous proofs see Carlitz, same Monthly 60, 181 (1953); these Rev. 14, 642; Popov, Bull. Soc. Math. Phys. Macédoine 4, 5–6 (1954); these Rev. 15, 847; and the reviewer, Amer. Math. Monthly 60, 179–181 (1953); these Rev. 14, 642]. *E. Grosswald* (Philadelphia, Pa.).

Yamamoto, Koichi. Euler squares and incomplete Euler squares of even degrees. Mem. Fac. Sci. Kyūsyū Univ. A. 8, 161–180 (1954).

The author, unaware of the reviewer's previous work, proves a theorem of the reviewer [Bull. Amer. Math. Soc. 50, 249–257 (1944), Th. I; these Rev. 6, 14]. He applies this theorem to the proof of the non-existence of an Euler square of side $4n+2$. This application of his theorem was also mentioned by the reviewer. The author also studies incomplete Euler squares with only two pairs of numbers missing and gives a systematic method for constructing such squares. *H. B. Mann* (Columbus, Ohio).

Okamoto, Masashi. On a certain type of matrices with an application to experimental design. Osaka Math. J. 6, 73–82 (1954).

The author considers matrices $A = (a_{ij})$ which can be completely reduced by a permutation. If furthermore the conditions $A = A^*$, $a_{ij} \geqq 0$, $a_{ii} = \sum_j' a_{ij} > 0$ $(i, j = 1, 2, \cdots)$ are satisfied, A is positive semi-definite. Such matrices occur in the theory of multiple correlation.

The author applies his results to a two-way classification design with unequal numbers of observations in each subclass. He shows that all row and column effects of such a design can be estimated if and only if the replication matrix cannot be completely reduced by permutations of rows and columns. *J. L. Brenner* and *H. B. Mann.*

Pagni, Mauro. Alcune osservazioni sui sistemi di m equazioni lineari ad n incognite. Boll. Un. Mat. Ital. (3) 9, 81–88 (1954).

Following ideas of G. Fichera [Lezioni sulle trasformazioni lineari, vol. 1, Ist. Mat., Univ., Trieste, 1954], the author develops the standard theorems for the existence and multiplicity of solutions of m equations in n unknowns without determinants or matrices. The role of rank of the coefficient matrix is played by the "characteristic" of the set of row vectors, i.e., the maximum number of linearly independent rows. *G. E. Forsythe* (Los Angeles, Calif.).

★**Taussky, Olga. Generalized commutators of matrices and permutations of factors in a product of three matrices.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 67–68. Academic Press Inc., New York, 1954. $9.00.

The following is proved: If R, S are any two elements of a group such that the product RS^{-1} is a commutator $CDC^{-1}D^{-1}$, and if (t, u, v) is any non-cyclic permutation of $(1, 2, 3)$, then elements A_1, A_2, A_3 can be found in the group such that the relations (*) $R=A_tA_uA_v$, $S=A_1A_2A_3$ are satisfied. The three sets of solutions are: $A_1=D$, $A_2=C$, $A_3=C^{-1}D^{-1}S$; $A_1=C^{-1}D^{-1}S$, $A_2=S^{-1}D$, $A_3=CS$; $A_1=CDS$, $A_2=S^{-1}D^{-1}S$, $A_3=S^{-1}C^{-1}S$; the corresponding (t, u, v) are $(2, 1, 3)$, $(3, 2, 1)$, $(1, 3, 2)$ respectively. Existence of A_1, A_2, A_3 is in each case a sufficient condition that RS^{-1} be a commutator. By a theorem of Shoda [Jap. J. Math. 13, 361–365 (1937)] every matrix of complex elements with determinant 1 is a commutator so that whenever R, S have the same determinant, A_1, A_2, A_3 can be found to satisfy the relations (*). *J. L. Brenner* (Aberdeen, Md.).

Fan, Ky. Some remarks on commutators of matrices. Arch. Math. 5, 102–107 (1954).

The following generalisation of O. Taussky's factorisation theorem [see the preceding review] is established: Two elements x, y of a group G can be expressed in the form $x=a_1a_2\cdots a_{2n+1}$, $y=a_{2n+1}\cdots a_2a_1$ $(n\geqq 1, a_i \in G)$, if and only if xy^{-1} is a product of n commutators of G; moreover, x, y can be expressed in the form $x=a_1a_2\cdots a_{2n}$, $y=a_{2n}\cdots a_2a_1$ $(n\geqq 1, a_i\in G)$, if and only if there exist $n-1$ commutators $c_1, \cdots, c_{n-1}$ and an element d of G such that $x=c_1\cdots c_{n-1}d^{-1}yd$. Analogues of the theorems of K. Shoda [Jap. J. Math. 13, 361–365 (1937)] and Taussky are obtained for the unitary group. Further results concern commutators composed of normal or Hermitian matrices. *D. E. Rutherford* (St. Andrews).

Sherman, S. A correction to "On a conjecture concerning doubly stochastic matrices." Proc. Amer. Math. Soc. 5, 998–999 (1954).

See same Proc. 3, 511–513 (1952); these Rev. 14, 346.

Amante, S. Funzioni di matrici. Matematiche, Catania 8, no. 2, 19–20 (1953).

Amante, S. Equazioni polidrome fra matrici. Matematiche, Catania 8, no. 2, 21–27 (1953).

For the sake of brevity the reviewer has altered the notation slightly. Let $f(z)$ be a many valued analytic function of z. M. Cipolla [Rend. Circ. Mat. Palermo 56, 144–154 (1932); MacDuffee, The theory of matrices, Springer, Berlin, 1933, p. 101] has defined the general function $f^*(X)$ of a matrix X with complex elements. This definition is not entirely satisfactory and in the first note the author suggests an improvement. The problem in the second paper is to solve the matrix equation $f^*(X)=A$. This is equivalent to finding the most general canonical matrix C such that $f^*(C)$ is similar to the canonical form B of A. Using $\dot{\sum}$ to denote a direct sum of submatrices, $f^*(C)=\dot{\sum}_{\mu,\epsilon} f(C_{\mu,\epsilon})$, where on the right hand side any determination of f for each $C_{\mu,\epsilon}$ is permissible, where $C=\dot{\sum} C_{\mu,\epsilon}$, $C_{\mu,\epsilon}=\rho_{\mu,\epsilon}I+N$, $I=(\delta_{r,s})$, $N=(\delta_{r+1,s})$. It is shown that $f(C_{\mu,\epsilon})$ must be similar to a direct sum $\dot{\sum}_k B_{\mu,k}$, where $B_{\mu,k}=\alpha_\mu I+N$, though k is not in general summed over all submatrices of B with a common latent root α_μ. This reduces the problem to one solved by the author [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 17, 31–36, 431–436 (1933)]. The statement of the results is rather lengthy for reproduction here. Somewhat similar results were obtained by H. Richter [Math. Ann. 122, 16–34 (1950); these Rev. 12, 235]. *D. E. Rutherford* (St. Andrews).

Gautschi, Werner. Bounds of matrices with regard to an Hermitian metric. Compositio Math. 12, 1–16 (1954).

Let A be a matrix of complex elements with m rows and n columns. Let $H>0$, $K>0$ be Hermitian positive definite matrices of orders m, n. For a vector z, define $\|z\|_H^2=z^*Hz$. The author introduces upper and lower bounds as follows:

$$\Omega_{H,K}(A)=\max_{\|x\|_K=1}\|Ax\|_H;\quad \omega_{H,K}(A)=\min_{\|x\|_K=1}\|Ax\|_H.$$

This paper, which is part of a Basle doctoral thesis, develops the properties of these bounds, paralleling a similar development for ordinary bounds $(H=I_m, K=I_n)$ by A. Ostrowski [Leçons sur la résolution des systèmes d'équations, Gauthiers-Villars, Paris, to appear].

The following are typical properties: If $m=n$ and A^{-1} exists, then $\Omega_{H,K}(A)\omega_{K,H}(A^{-1})=1$. If S, T are regular, then $\Omega_{H,K}(A)=\Omega_{S^*HS,T^*KT}(S^{-1}AT)$ and the same for $\omega_{H,K}$. Theorem: $\Omega_{K,H}(A^*)=\Omega_{H^{-1},K^{-1}}(A)$ and, if $m=n$,

$$\omega_{K,H}(A^*)=\omega_{H^{-1},K^{-1}}(A).$$

A series of five examples illustrate the bounds (here denoted by Ω_H, ω_H) when $m=n$ and $H=K$.

The author generalizes a theorem of W. Ledermann [J. London Math. Soc. 12, 12–18 (1937)], and applies this to study when $\inf_{H>0}\Omega_H(A)=|\lambda_A|^{\max}$ $(\sup_{H>0}\omega_H(A)=|\lambda_A|^{\min})$ is attained for suitable H. (Here λ_A denotes an eigenvalue of A.) The inf (sup) is attained if and only if the elementary divisors associated with all the maximal (minimal) eigenvalues of A are simple. *G. E. Forsythe.*

★**Fan, Ky, and Hoffman, A. J. Lower bounds for the rank and location of the eigenvalues of a matrix.** Contributions to the solution of systems of linear equations and the determination of eigenvalues, pp. 117–130. National Bureau of Standards Applied Mathematics Series No. 39. U. S. Government Printing Office, Washington, D. C., 1954. $2.00.

The problems considered are: i) find lower bounds for the rank of an n by n matrix A with complex elements that can be calculated in a simple manner from the coefficients; and, ii) find nonnegative numbers $\rho_1, \cdots, \rho_n$ such that every eigenvalue lies in at least one of the n circular disks $|\lambda-a_{ii}|\leqq\rho_i$. Results in the cited literature are stated or reinterpreted in terms of these problems. Extensions or new results obtained include the following. If rank $A=r$, then 1) $r\geqq m$ if, for at least m distinct indices i, $|a_{ii}|\geqq$ the maximum sum of the moduli of $m-1$ distinct off-diagonal elements of the ith row of A; 2) $r\geqq\alpha_1/\beta_1+\cdots+\alpha_s/\beta_s$, where 0/0 is interpreted as 0, $A=(A_{pq})$ with A_{pq} submatrices $(p, q=1, 2, \cdots, s)$, $\alpha_p=\text{norm}^2 A_{pp}$ and $\beta_p=\sum_q \text{norm}^2 A_{qp}$; and,

$$r\geqq\sum_i\{|a_{ii}|\div[|a_{ii}|+\alpha_i^{1/q}(\sum_{j\neq i}|a_{ij}|^p)^{1/p}]\},$$

where $\alpha_i>0$, $\sum_{i=1}^n(1+\alpha_i)^{-1}\leqq 1$, $p>1$ and $p^{-1}+q^{-1}=1$. The radii ρ_i of the disks of problem ii) may be taken to be $\alpha(\sum_{j\neq i}|a_{ij}|^p)^{1/p}$ provided

$$\alpha^q(1+\alpha^q)\geqq\sum_i(\sum_{j\neq i}|a_{ij}|)^q\div(\sum_{j\neq i}|a_{ij}|^p)^{q/p}.$$

If λ is a root of multiplicity s, it must lie in one of the n disks of ii) for $\rho_i=\beta_i^{\frac{1}{2}}(\sum_{j\neq i}|a_{ij}|^2)^{\frac{1}{2}}$, where $\beta_i>0$ and

$\sum_i(1+\beta_i)^{-1}\leqq s$. Alternatively, $\rho_i=\beta_i\sum_{j\neq i}|a_{ij}|$ under the same conditions. For normal matrices inequalities are derived for subsets of the eigenvalues from maximum and minimum characterizations of such subsets. If x and y are vectors in unitary m and n dimensional spaces, M and N are normal $m\times m$ and $n\times n$ matrices and $(x,x)=(y,y)>0$, $(Mx,x)=(Ny,y)$, $(Mx,Mx)>(Ny,Ny)$, then every closed circular disk containing all the eigenvalues of M contains at least one of N. *W. Givens* (Princeton, N. J.).

***Fan, Ky. Inequalities for eigenvalues of Hermitian matrices.** Contributions to the solution of systems of linear equations and the determination of eigenvalues, pp. 131–139. National Bureau of Standards Applied Mathematics Series No. 39. U. S. Government Printing Office, Washington, D. C., 1954. $2.00.

The eigenvalues λ_i of an $n\times n$ Hermitian matrix $H=(a_{ij})$ are proved to lie in the intervals $-d_{n-i+1}\leqq\lambda_i\leqq d_i$ if the n d_i are non-negative and $n-1$ numbers $c_i>1$ can be found such that $d_i(c_i^2-1)\geqq c_i^2d_{i+1}$, $i=1,\cdots,n-1$, and $|a_{ii}|+c_i(\sum_{j>i}|a_{ij}|^2)^{1/2}\leqq d_i$, $i=1,\cdots,n$. Requiring only d_i real, $c_i>0$, $d_i-d_{i+1}\geqq c_i^{-1}$ and $a_{ii}+c_i\sum_{j>i}|a_{ij}|^2\leqq d_i$ still yields $\lambda_i\leqq d_i$. If $H=(A_{\alpha\beta})$ for block matrices $A_{\alpha\beta}$ has eigenvalues $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n$ and the direct sum matrix $K=A_{11}\oplus A_{22}\oplus\cdots\oplus A_{s+1,s+1}$ has eigenvalues $\kappa_1\geqq\cdots\geqq\kappa_n$, then $\sum_{i=1}^h\kappa_i\leqq\sum_{i=1}^h\lambda_i$; for H positive definite,

$$\prod_{i=1}^h\lambda_{n-i+1}\leqq\prod_{i=1}^h\kappa_{n-i+1},\quad 1\leqq h\leqq n.$$

If $A_{\alpha\beta}$ has $p_\alpha-p_{\alpha-1}$ rows and $p_\beta-p_{\beta-1}$ columns ($p_0=0$), $b_i=(\sum_{j>p_k}|a_{ij}|^2)^{1/2}$, where $p_{k-1}<i\leqq p_k$, $\beta_1\geqq\beta_2\geqq\cdots\geqq\beta_{p_s}$ are the b_i rearranged and $\gamma_i=\beta_i+\beta_{i+1}+\cdots+\beta_{p_s}$ ($1\leqq i\leqq p_s$), then $\lambda_i\leqq\kappa_j+\gamma_{i-j+1}$ if $1\leqq i-j+1\leqq p_s$, $\kappa_j-\gamma_{j-i+1}\leqq\lambda_i$ if $1\leqq j-i+1\leqq p_s$, $\lambda_i\leqq\kappa_{i-p_s}$ if $i-p_s\geqq 1$, $\lambda_i\geqq\kappa_{i+p_s}$ if $i+p_s\leqq n$, and $\sum_{i=1}^h(\lambda_i-\kappa_i)\leqq\gamma_1$ ($1\leqq h\leqq n$). *W. Givens.*

Veltkamp, G. W., Duparc, H. J. A., and Peremans, W. A minimum problem on matrices. Math. Centrum Amsterdam. Rapport **ZW 1954-007**, 2 pp. (1954).

Let C be a constant positive definite $m\times n$ matrix, p a constant $m\times k$ matrix of rank k; E a constant $k\times 1$ matrix. For various L (a $k\times m$ matrix), what is the minimum value of

$$\tau^2=E^*(LP)^{-1}LCL^*(P^*L^*)^{-1}E?$$

The answer is $E^*(P^*C^{-1}P)^{-1}E$, attained for $L=P^*C^{-1}$.
J. L. Brenner (Aberdeen, Md.).

Wu, T. T. On the perturbation of characteristic vibrations by dissipative forces. Arch. Math. **5**, 175–181 (1954).

Following a technique given by Rosenbloom [cf. Bull. Amer. Math. Soc. **58**, 483–484 (1952)], the author demonstrates existence and uniqueness and provides numerical estimates for the solution of the following problem. Consider the equation (*) $(p^2I+\Lambda+pW)\xi=0$, where ξ is a vector in unitary n-space, p is a complex number, I is the identity matrix, Λ is a given diagonal matrix $=\operatorname{diag}(\lambda_1^2,\cdots,\lambda_n^2)$ with $\lambda_i\geqq 0$ and W is a given matrix of small norm. For a preassigned r it is assumed that $\lambda_r>0$ and $\lambda_j\neq\lambda_r$ for $j\neq r$. Then ξ and p are to be found such that: (*) is satisfied, ξ is near to the normalized solution x of the equation $\Lambda x=\lambda_r^2x$, and p is near $i\lambda_r$. The problem arises immediately in the consideration of the perturbations of the characteristic vibrations of a linear system by small damping forces proportional to the velocity. *H. D. Block.*

Egerváry, E. On hypermatrices whose blocks are commutable in pairs and their application in lattice-dynamics. Acta Sci. Math. Szeged **15**, 211–222 (1954).

This paper duplicates a paper by Afriat [Quart. J. Math., Oxford Ser. (2) **5**, 81–98 (1954); these Rev. **16**, 105], and obtains in addition the characteristic vectors of a symmetric matrix, the elements of the matrix being themselves pairwise commutative square matrices. As an application, the author gives a rigorous analysis of the problem of normal vibrations of two- and three-dimensional lattices of particles with fixed boundary. The physico-theoretic results obtained are analogous to those of M. Born [Vorlesungen über Krystalldynamik, Springer, 1913] and to the recent book of L. Brillouin [Wave propagation in periodic structures, 2nd ed., Dover, New York, 1953; these Rev. **14**, 704].
J. L. Brenner and *H. McIntosh* (Aberdeen, Md.).

Egerváry, E. On the contractive linear transformations of n-dimensional vector space. Acta Sci. Math. Szeged **15**, 178–182 (1954).

The author shows, by means of an explicit geometric-matricial computation, that if T is a contraction (i.e., $\|T\|\leqq 1$) of complex n-dimensional Euclidean space, then there exists a unitary transformation U on a space of dimension $(k+1)n$ such that T^i is the compression (northwest corner) of U^i for $i=1,\cdots,k$. *P. R. Halmos.*

Egerváry, J. Auf dyadischer Matrizendarstellung beruhende Methode zur Transformation bilinearer Formen und Auflösung linearer Gleichungsysteme. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 11–32 (1954). (Hungarian. Russian and German summaries)

A partly historical and mostly expository treatment. Some numerical examples are worked out.
P. R. Halmos (Chicago, Ill.).

Klingenberg, Wilhelm. Paare symmetrischer und alternierender Formen zweiten Grades. Abh. Math. Sem. Univ. Hamburg **19**, no. 1–2, 78–93 (1954).

Let A and B be a pair of symmetric or skew matrices with elements in an arbitrary field K of characteristic different from 2. Call A and B a regular pair if there are numbers a and b in K such that $\det(aA+bB)\neq 0$. Using methods of H. Weyl and O. Schreier, the author shows that if A, B is a regular pair of symmetric or skew matrices, then, to a decomposition of the characteristic polynomial $\det(A^{-1}B-xI)$ into s prime powers, corresponds, by a transformation of coordinates, a decomposition of A and B into the direct sum of s matrices A^i and B^i. If A and B are symmetric and the prime power factors of the characteristic equation are powers of linear factors, it is shown that each A^i is the direct sum of matrices which are scalar multiples of matrices C whose elements on the non-principal diagonal are 1 and all other elements 0; while each B^i is a scalar multiple of a_iC+U, where U is the matrix whose only non-zero elements are 1's immediately below the non-principal diagonal. Similar forms are obtained for the skew matrices. The expression as a direct sum is unique for the skew case and, if certain constants are squares in K, unique for the symmetric case as well. Pairs A, B are also considered in which the rank of the matrix $xA+yB$ in indeterminates x and y is less than the order, the so-called singular case. Similar results have been found for fields of characteristic zero by J. Williamson [Amer. J. Math. **57**, 475–490 (1935); **59**, 399–413 (1937)]. *B. W. Jones* (London).

Oppenheim, A. Inequalities connected with definite Hermitian forms. II. Amer. Math. Monthly **61**, 463–466 (1954).

[For part I see J. London Math. Soc. **5**, 114–119 (1930).] If A and B are positive semi-definite Hermitian matrices and $C=A+B$ with respective characteristic values $\alpha_1\leqq\alpha_2\leqq\cdots\leqq\alpha_n$, $\beta_1\leqq\beta_2\leqq\cdots\leqq\beta_n$ and $\gamma_1\leqq\gamma_2\leqq\cdots\leqq\gamma_n$, then $(\gamma_1\gamma_2\cdots\gamma_i)^{1/i}\geqq(\alpha_1\alpha_2\cdots\alpha_i)^{1/i}+(\beta_1\beta_2\cdots\beta_i)^{1/i}$ for $i=1, 2, \cdots, n$. The first proof given is incorrect since it requires A and B to be simultaneously carried into diagonal form and this will, unless A and B commute, require a non-orthogonal congruence, not preserving the characteristic values of C. The second proof given is correct.

An algebraic inequality correctly established in the course of the first proof can be written:

$$(e_1+f_n)(e_2+f_{n-1})\cdots(e_n+f_1)\geqq\prod_i(e_i+f_{p(i)})\geqq\prod_i(e_i+f_i),$$

where $0\leqq e_1\leqq e_2\leqq\cdots\leqq e_n$, $0\leqq f_1\leqq\cdots\leqq f_n$ and $i\to p(i)$ is any permutation of $1, 2, \cdots, n$. *W. Givens.*

Iseki, Kanesiroo. On the fundamental theorem of algebra. J. Math. Soc. Japan **6**, 129–130 (1954).

The theorem of the title is proved with the aid of the theory of normed rings. *M. Henriksen.*

Stöhr, Alfred. Eine Formel für gewisse symmetrische Polynome. Acta Sci. Math. Szeged **15**, 209–210 (1954).

Let x_i be letters; let σ_i be the ith elementary symmetric function of them ($i=1, \cdots, n$; $\sigma_0=1$). Let a_i be integers ($0\leqq a_1<a_2<\cdots<a_n$). Let c_{ij} be $x_j^{a_i}$; let d_{ij} be x_j^i. Let m satisfy $m\geqq\max(1, a_n-n+1)$. Let integers b_i be so determined ($b_1<b_2<\cdots<b_m$) that the set $\{a_1, \cdots, a_n, b_1, \cdots, b_m\}$ is the set $\{0, \cdots, n+m-1\}$. Then the relation

$$\det(c_{ij})_1^n/\det(d_{ij})_1^n=\sum\pm\prod_{t=1}^m\sigma_{n+c_t-b_t}$$

holds. Here $\sum$ is extended over all permutations $[c_t]_1^m$ of $[t-1]_1^m$ for which the relations $0\leqq b_t-c_t\leqq n$ hold; and $\pm$ is the sign of the corresponding permutation $[c_t]$. This formula generalizes that of Polya and Szegö [Aufgaben und Lehrsätze aus der Analysis, vol. 2, Springer, Berlin, 1925, sec. VII, no. 10, pp. 99, 302]. *J. L. Brenner.*

Fadini, Angelo. L'algebra A somma diretta di k algebre n_i-duali. Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 281–286 (1952).

Ibrahim, E. M. Note on a paper by Murnaghan. Proc. Nat. Acad. Sci. U. S. A. **40**, 1000–1001 (1954).

Murnaghan [same Proc. **37**, 439–441 (1951); these Rev. **13**, 201] has calculated the plethysm of S-functions $\{8\}\otimes\{3\}$ by using the theorem that the terms beginning with n in the expansions of either $\{1^m\}\otimes\{1^n\}$ or $\{1^m\}\otimes\{n\}$ are obtained by prefixing n to the terms of the expansions $\{1^{m-1}\}\otimes\{1^n\}$ or $\{1^{m-1}\}\otimes\{n\}$ respectively. The author points out that this theorem is a particular case of his own theorem II [Quart. J. Math., Oxford Ser. (2) **3**, 50–55 (1952); these Rev. **14**, 243] that the principal parts of the products appearing in the expansion

$$[\{\lambda_1, \cdots, \lambda_r\}\otimes\{1^n\}][\{\mu_1, \cdots, \mu_r\}\otimes\{1^n\}]$$

appear in

$$\{\lambda_1+\mu_1, \cdots, \lambda_r+\mu_r\}\otimes\{n\}.$$

He gives a shorter method of calculating $\{8\}\otimes\{3\}$ using the principal parts of

$$[\{6\}\otimes\{3\}][\{2\}\otimes\{3\}] \quad\text{and}\quad [\{6\}\otimes\{1^3\}][\{2\}\otimes\{1^3\}].$$

The expansions $\{6\}\otimes\{3\}$ and $\{6\}\otimes\{1^3\}$ can be found from tables calculated by the author showing $\{\lambda\}\otimes\{\mu\}$ for all partitions up to a total degree 18 in the resultant. These tables are available on request from the Royal Society, London. The author gives also a method of determining certain S-functions which are common to the expansions $\{8\}\otimes\{3\}$ and $\{3\}\otimes\{8\}$. *D. E. Littlewood* (Bangor).

Duncan, D. G. Note on the algebra of S-functions. Canadian J. Math. **6**, 509–510 (1954).

Representing $\{\mu\}\otimes S_i$ by t_i, the author obtains the formula

$$\begin{aligned}\{\mu\}\otimes\{2k\}=(\{\mu\}\otimes\{2k-1\})\{\mu\}\\ -(\{\mu\}\otimes\{2k-2\})(\{\mu\}\otimes\{2\})+\cdots\\ +\frac{(-1)^{k+1}}{2}(\{\mu\}\otimes\{k\})^2+\tfrac{1}{2}\sum_{(\beta)}\frac{1}{\beta_1!\cdots\beta_k!}\left(\frac{t_2}{1}\right)^{\beta_1}\cdots\left(\frac{t_{2k}}{k}\right)^{\beta_k},\end{aligned}$$

which he suggests as a shorter way of computing $\{\mu\}\otimes\{2k\}$. *D. E. Littlewood* (Bangor).

Abstract Algebra

★Dubreil, Paul. Algèbre. Tome I. Equivalences, opérations, groupes, anneaux, corps. 2ème éd. Gauthier-Villars, Paris, 1954. 467 pp. 3900 francs.

The various topics treated in the first edition [1946; these Rev. **8**, 192] have been brought up to date in this new edition. Also, some new topics have been added. The first chapter now has a complete discussion of relations, as well as a new section on ordered sets and lattices. The fifth chapter on ordered fields has been enlarged to include more material on the imbedding problem for demi-groups. *R. E. Johnson* (Northampton, Mass.).

★Zappa, Guido. Gruppi, corpi, equazioni. 2a ed. A cura del Prof. Rodolfo Permutti. Libreria Editrice Liguori, Napoli, 1954. 312 pp.

The present text contains a concise and very careful presentation of the basic parts of group theory, the theory of fields and the Galois theory of equations. The greater part of the book is devoted to group theory. The two first chapters give the principal group concepts. In the third one finds a discussion of lattices and the chain theorems of Jordan-Hölder and Schreier-Zassenhaus, followed by a more advanced study of non-normal chains, permutable chains and arbitrary maximal chains. Subsequently one finds the basis theorem for Abelian groups with a finite number of generators, the Sylow theorems, p-groups and a few results by P. Hall on central series. There follows a section on the main properties of rings and field. In the last chapters one finds an exposition of the general Galois theory following in the main the lines suggested by Artin. *O. Ore.*

Rainich, G. Y. Involution and equivalence. Michigan Math. J. **2**, 33–34 (1954).

On peut définir une involution comme une relation binaire $(xy)^*$ satisfaisant aux deux axiomes: A) $(xy)^*$ implique $(yx)^*$; B) $(zx)^*$, $(xy)^*$, $(yu)^*$ implique $(zu)^*$. D'ou une analogie avec les relations d'équivalence. *J. Riguet.*

Sholander, Marlow. Medians and betweenness. Proc. Amer. Math. Soc. **5**, 801–807 (1954).

L'auteur a montré récemment [mêmes Proc. **3**, 369–381 (1952); ces Rev. **14**, 9] que les semi treillis à médiane constituent une généralisation commune aux treillis et aux arbres. Il donne ici une caractérisation de ces semi treillis particuliers au moyen du concept de segment, puis au moyen de la relation ternaire "être situé entre". Il démontre qu'on peut caractériser les treillis distributifs au moyen de la notion de médiane. *J. Riguet* (Paris).

Sholander, Marlow. Medians, lattices, and trees. Proc. Amer. Math. Soc. **5**, 808–812 (1954).

Un ensemble est appelé semi treillis s'il est muni d'une multiplication associative, commutative, idempotente [G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 18; ces Rev. **10**, 673]. On considère quatre postulats L_0, L_1, L_2, L_3 dont la liste se trouve au §1. Alors un semi treillis à médiane, respectivement un arbre, un treillis distributif, une chaine peuvent être caractérisés comme semi treillis dont les multiplications satisfont respectivement à L_0, L_1, L_2, L_3. On montre, en outre, qu'un semi treillis distributif peut toujours être immergé dans un treillis distributif. *J. Riguet* (Paris).

Wooyenaka, Yuki. On Newman algebra. I. Proc. Japan Acad. **30**, 170–175 (1954).

A non-empty system K closed under binary operations of addition and multiplication, with a unary operation $a\to a'$ (not assumed single-valued) and satisfying $a+b=b+a$, $a(b+c)=ab+ac$, $a+b'b=a$, $a(b'+b)=a$ and the cyclic associative law $a(bc)=b(ca)$ is a direct union of a Boolean lattice and an associative Boolean ring with unity. Adding the axiom $a+a=a$ characterizes the lattice and adding either $a+a=0$ or $(a'+a)+a=a'$ the ring. The novelty of the axioms lies in the implied associativity. Independence examples are given. *W. Givens* (Princeton, N. J.).

Wooyenaka, Yuki. On Newman algebra. II. Proc. Japan Acad. **30**, 562–565 (1954).

[See the preceding review.] The commutative law for addition is weakened to $a+b'b=b'b+a$, and a new proof of the associativity of addition and new independence examples are given. *W. Givens* (Princeton, N. J.).

Lesieur, Léonce. Sur l'algèbre de la topologie. C. R. Acad. Sci. Paris **238**, 1464–1466 (1954).

The neighborhood definition of a topology on a set E is expressed in terms of the lattices $L(E)$ of subsets of E, and $\Phi(L)$ of closed elements of L, thus leading to definitions of a topology for an abstract complete Boolean lattice L, or distributive lattice (with 0 and 1) Φ. A topology on L is defined to be a mapping V into the lattice of filters (dual ideals) such that (where (a) denotes principal filter): (1) $(a)\supset V(a)$, $(0)=V(0)$; (2) $V(\bigcup a)=\bigcap V(a)$; (3) $v\in V(a)$ implies existence of b with $v\in V(b)$, $b\in V(a)$. Interpretation: $v\in V(a)$ if $a\subset u\subset v$ for some open u. A topology of closed neighborhoods on Φ is defined by $v\in V(a)$ if $a\cap x=0$, $x\cup v=1$, for some x. Then V satisfies (1) and the finite case of (2). Separation is defined for elements of Φ. A Kuratowski lattice is a complete distributive Φ in which every element is a union of points; this corresponds to $\Phi=\Phi(L(E))$ for E a Kuratowski T_1-space. Separation axioms for Φ are given corresponding to T_2, T_3, T_4 on E. *R. C. Lyndon.*

Sampei, Yoemon. Supplement to the paper entitled "On lattice completions and closure operations". Comment. Math. Univ. St. Paul. **3**, 29–30 (1954).

See same journal **2** (1953), 55–70; these Rev. **15**, 675.

Riera, E. Lluis, and Recillas Juárez, F. On prime ideals in generalized semilocal rings. Bol. Soc. Mat. Mexicana **10**, nos. 3–4, 19–22 (1953). (Spanish)

Soit $\mathfrak{o}$ un anneau de Zariski, c.à.d. un anneau noethérien topologisé par les puissances $\mathfrak{m}^n$ d'un idéal $\mathfrak{m}$ contenu dans l'intersection des idéaux maximaux de $\mathfrak{o}$. Si $\mathfrak{p}$ est un idéal maximal de $\mathfrak{o}$, alors $\hat{\mathfrak{o}}\cdot\mathfrak{p}$ est un idéal maximal du complété $\hat{\mathfrak{o}}$ de $\mathfrak{o}$, et il y a correspondance biunivoque entre les idéaux primaires pour $\mathfrak{p}$ et pour $\hat{\mathfrak{o}}\cdot\mathfrak{p}$. *P. Samuel.*

Nagata, Masayoshi. Note on complete local integrity domains. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 271–278 (1954).

En se servant de la notion de produit tensoriel (complété) d'anneaux locaux complets, l'auteur étend aux anneaux d'intégrité de la forme $k[[x_1, \cdots, x_n]]$ diverses propriétés des anneaux d'intégrité de type fini $k[x_1, \cdots, x_n]$ [ou plutôt de leurs corps des fractions; cf. Weil, Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., vol. 29, New York, 1946; ces Rev. **9**, 303]. Ce sont les propriétés utiles dans l'étude de l'extension du corps de base et dans celle des variétés produits: séparabilité, dérivations et critères de séparabilité au moyen des dérivations, extensions régulières, produits tensoriels d'extensions régulières.
P. Samuel (Clermont-Ferrand).

Tominaga, Hisao. On right-regular-ideal-rings. Proc. Japan Acad. **29**, 486–489 (1953).

Let K be a ring with a unit element and let M be a unitary K-right module. Then M is said to be right regular, if there exists a pair u, v of natural integers such that the direct sum of u copies of M is a free K-module with v generators. The ring K is called a right regular ideal ring, if every non-zero right ideal of K is right regular. The author proves a number of simple remarks. For instance, it is shown that if K is a right regular ideal ring and M a right regular K-module, every non-zero submodule of M is right regular.
R. Brauer (Cambridge, Mass.).

Kasch, Friedrich. Über den Automorphismenring einfacher Algebren. Arch. Math. **6**, 59–65 (1954).

Let A be the set of $n\times n$ matrices ($n>1$) over a field Z. Considering A merely as a vector space over Z, the group of inner automorphisms G of A acts as a group of linear transformations on A, as does multiplication by elements of Z. Let R be the ring of linear transformations generated by Z and G. Let B be the additive subgroup of A generated by all commutators $ab-ba$. The author proves the following. If A is the $n\times n$ matrix algebra over Z, and Z is not the prime field of two elements, and if n is prime to the characteristic of Z, then 1) A, as an R module, is completely decomposable and has a direct sum decomposition in irreducible components as $A=Z\oplus B$, 2) R induces all linear transformations of B over Z, 3) R is semi-simple and is a direct sum of Z and the set of all (n^2-1) by (n^2-1) matrices over Z. The proof is fairly obvious and computational in nature. *I. N. Herstein* (Philadelphia, Pa.).

Beyer, Gudrun. Ein Einzigkeitssatz in der Einbettungstheorie galoisscher Körper. Math. Nachr. 11, 317–320 (1954).

The paper gives a supplement to an investigation of P. Wolf on Galois algebras [Math. Nachr. 9, 281–300 (1953); these Rev. 15, 97]. The existence of a solution of the imbedding problem treated there depended on the existence of a solution of certain equations. In the present paper, it is shown how the most general solution of the latter equations can be obtained, if one solution is known. In the case of abelian algebras, the corresponding result had already been given by H. Hasse [ibid. 1, 40–61 (1948); these Rev. 10, 426]. *R. Brauer* (Cambridge, Mass.).

Wolf, Paul. Die direkte Zerlegung verallgemeinerter galoisscher Algebren mit Einselement und die Multiplikation verallgemeinerter abelscher Algebren. Abh. Math. Sem. Univ. Hamburg 19, no. 1–2, 94–114 (1954).

The author continues his investigations on generalized Galois algebras K over a ground field Ω with a Galois group $\mathfrak{G}$ [see, in particular, Math. Nachr. 9, 201–216 (1953); these Rev. 15, 6]. A necessary and sufficient condition is given that such an algebra possesses a unit element. The decomposition of K into a direct sum of indecomposable algebras L_i is studied. All these L_i are isomorphic. Their number is equal to the index of a certain subgroup $\mathfrak{G}^*$ of $\mathfrak{G}$. In the last part of the paper, the author considers the system $\mathfrak{A}$ of generalized abelian algebras over Ω with a fixed Galois group $\mathfrak{G}$. The product of two elements of $\mathfrak{A}$ is defined as an element of $\mathfrak{A}$. This multiplication corresponds to the multiplication of the classes of normal factors corresponding to the algebras [see the paper cited above]. Then $\mathfrak{A}$ is a semigroup. The subsets of algebras in $\mathfrak{A}$ with unit element and of commutative algebras are both closed under multiplication. The separable algebras in $\mathfrak{A}$ form a subgroup $\mathfrak{S}$. Every element of $\mathfrak{S}$ has finite order. *R. Brauer.*

Osima, Masaru. Some studies on Frobenius algebras. II. Math. J. Okayama Univ. 3, 109–119 (1954).

The first part appeared in Jap. J. Math. 21, 179–190 (1952); these Rev. 14, 841. The author proves a number of known results including some of T. Nakayama [Proc. Japan Acad. 25, no. 7, 45–50 (1949); these Rev. 12, 797] and of M. Ikeda [Osaka Math. J. 3, 227–239 (1951); these Rev. 13, 719]. Further results deal with the basic algebras of Frobenius algebras, symmetric, almost symmetric and weakly symmetric algebras. *R. Brauer.*

Jacobson, N. Structure of alternative and Jordan bimodules. Osaka Math. J. 6, 1–71 (1954).

In these pages the author develops a comprehensive account of the theory of bi-representations of a class of alternative and Jordan algebras using new methods, the study of involutions in matrix algebras determined by involutions in the coefficient algebra, to extend and complete his own and Schafer's earlier work. The methods, which do not depend on the theory of Lie algebras, remove restrictions of characteristic, give information about certain infinite-dimensional algebras, and also serve to determine the irreducible bi-representations of finite-dimensional separable algebras.

To be more specific let $\mathfrak{A}$ be the $n\times n$ matrix algebra over the (possibly non-associative) algebra $\mathfrak{D}$ with identity. Intrinsically $\mathfrak{A}$ is an algebra with a set of n^2 matrix units $\{e_{ij}\}$ in its nucleus and $\mathfrak{D}$ is the centralizer of $\{e_{ij}\}$ in $\mathfrak{A}$. Involutions in $\mathfrak{D}$ give rise to involutions, called standard, in $\mathfrak{A}$ which transpose a set of matrix units. For any algebra $\mathfrak{A}$ let $\mathfrak{A}_J$ denote the space $\mathfrak{A}$ equipped with the symmetrized product $\{xy\}=xy+yx$. If $a\to a^*$ is an involution $\mathfrak{A}_J$ has a subalgebra $\mathfrak{H}$ consisting of those elements self-adjoint with respect to the involution. If $\mathfrak{A}=\mathfrak{D}_n$ and $a\to a^*$ is standard the subalgebras of $\mathfrak{H}$ containing $\{e_{ii}\}$ and $\{e_{ij}+e_{ji}\}$ are the algebras of self-adjoint elements in matrix algebras. The homomorphisms of $\mathfrak{H}$ which preserve the associativity of $\{e_{ii}\}$, $\{e_{ij}+e_{ji}\}$ can all be extended to homomorphisms of $\mathfrak{A}$ if $n\geqq 3$. One may then ask for the conditions on $\mathfrak{D}$ under which $\mathfrak{H}$ is a Jordan algebra. For $n\geqq 3$ it turns out that, approximately, $\mathfrak{H}$ is Jordan if and only if either $n\geqq 4$ and $\mathfrak{D}$ is associative or $\mathfrak{D}$ is alternative and its self-adjoint elements are in the center. In any case $\mathfrak{H}$ is not special, i.e., isomorphic to a subalgebra of some associative $\mathfrak{B}_J$, $\mathfrak{B}$ associative, unless $\mathfrak{D}$ is associative. For those $\mathfrak{H}$ which are Jordan algebras the multiplication table for $\{e_{ii}\}$ and $\{e_{ij}+e_{ji}\}$ is characteristic. Thus if $\mathfrak{J}$ is a Jordan algebra with such a set of elements and $n\geqq 3$, then $\mathfrak{J}\cong\mathfrak{H}$ for some $\mathfrak{A}=\mathfrak{D}_n$.

These structural results can be put to work in the study of bi-representations. For Jordan algebras $\mathfrak{J}$ with identities there is an immediate reduction to those which are special and represent 1 by the multiple $\frac{1}{2}$ or those, called initial, which represent 1 by the identity transformation. These classes have universal associative covers $\mathfrak{U}_s$ and $\mathfrak{U}_1$ whose representations are in one-to-one correspondence with the special and unital bi-representations of $\mathfrak{J}$ respectively. $\mathfrak{U}_s$ can be used in the determination of $\mathfrak{U}_1$ since the sum of two commuting special bi-representations is unital and there is therefore a homomorphism of $\mathfrak{U}_1$ onto the Kronecker sum subalgebra $\mathfrak{U}_s^{(2)}$ of $\mathfrak{U}_s\otimes\mathfrak{U}_s$. The image subalgebra is the universal cover for sums of commuting special bi-representations. The algebras $\mathfrak{H}$ described above with $n\geqq 4$ have $\mathfrak{U}_s=\mathfrak{A}=\mathfrak{D}_n$ and $\mathfrak{U}_1\cong\mathfrak{U}_s^{(2)}$. For $n\geqq 3$ there is, between the equivalence classes of unital bi-representations of $\mathfrak{H}$ and those of $\mathfrak{D}$, a one-to-one correspondence which, as will be seen below, directs attention to the alternative bi-representations of three elementary types of associative algebras and the Cayley algebra.

For finite-dimensional separable algebras nearly complete results are obtained. A series of reductions shows the problems are essentially those for simple Jordan and alternative algebras over algebraically closed fields Φ. The former are known to be either (I) algebras $\mathfrak{J}$ of self-adjoint elements in certain matrix algebras $\mathfrak{A}=\mathfrak{D}_n$, $n\geqq 3$, or (II) algebras determined by a non-degenerate symmetric bilinear form in a vector space. In case II where $\{xy\}=(x,y)\cdot 1$, $\mathfrak{U}_s$ is the Clifford algebra of the form, $\mathfrak{U}_1$ is shown to be isomorphic to $\mathfrak{U}_s^{(2)}$ and semi-simple. The dimensions of its simple components are computed. In case I the possible coefficient rings are (A) $\mathfrak{D}=\Phi\oplus\Phi$ and $\mathfrak{J}\cong\Phi_{n,J}$, (B) $\mathfrak{D}=\Phi$, (C) $\mathfrak{D}=\Phi_2$, (E) $\mathfrak{D}=\mathfrak{C}$, the Cayley algebra, and $n=3$. In each case $\mathfrak{U}_1$ is semi-simple, its simple components are determined; and, since $\mathfrak{U}_s$ is semi-simple, each bi-representation is completely reducible. For example, in the exceptional case (E) the problem is to determine the unital bi-representations for $\mathfrak{C}$. These turn out to be direct sums of the regular bi-representation. It follows that the same is true for the exceptional simple Jordan algebra. *W. G. Lister.*

Seligman, George B. On a class of semisimple restricted Lie algebras. Proc. Nat. Acad. Sci. U. S. A. 40, 726–728 (1954).

A summary of the author's findings relating to the classification, over algebraically closed fields of characteristic $p>2$, of simple restricted Lie algebras which have restricted

representations with non-degenerate trace form. He reports that for $p>7$ the classes $A-D$ and analogues of the exceptional algebras are the only ones. The converse is not quite settled. *W. G. Lister* (Providence, R. I.).

Shanks, M. E., and Pursell, Lyle E. The Lie algebra of a smooth manifold. Proc. Amer. Math. Soc. 5, 468–472 (1954).

The algebra D of the infinitely differentiable real functions on an infinitely differentiable manifold X determines uniquely the structure of X apart from infinitely differentiable homeomorphisms [L. E. Pursell, Thesis, Purdue Univ., 1952; Shanks, Bull. Amer. Math. Soc. 57, 295 (1951)]. The Lie algebra L_0 of the tangent vector fields which map D into D_0 (functions with compact support) has the same property. X is constructed from L_0 by means of the maximal ideals. *H. Freudenthal* (Utrecht).

Theory of Groups

Stolt, Bengt. Weitere Untersuchungen zur Gruppenaxiomatik. Ark. Mat. 3, 89–101 (1954).

Dans sa thèse [Univ. of Uppsala, 1953, ces Rev. 15, 99] l'auteur avait donné des systèmes d'axiomes complets, d'autres incomplets et d'autres indéterminés. Il démontre ici que quatre systèmes sont complets et que les systèmes déduits sont irréductibles. Il montre que par introduction de deux nouveaux axiomes on peut déduire un système complet qui contient moins d'axiomes que le système du travail cité. *J. Riguet* (Paris).

Tsuboi, Teruo. On abelian normal subgroups. Sci. Rep. Saitama Univ. Ser. A. 1, 101–104 (1954).

If G is a finite group and A an abelian normal subgroup of G whose order is prime to its index, then A is the direct product of its intersections with the center and the derived group of G [Zassenhaus, Lehrbuch der Gruppentheorie, Teubner, Leipzig-Berlin, 1937, p. 136]. The author generalizes this result to the following theorem: If G is a finite group, not a p-group, and if A is an abelian normal subgroup of G whose index is a power of p, then A is the direct product of its intersections with the hypercenter of G and the $(c+1)$th term of the lower central series of G, where c is the class of the p-Sylow subgroups of G. The proof is obtained by writing A as the direct product of its intersection with an arbitrary p-Sylow subgroup of G and of the group B of all elements of A whose order is prime to p. The above result can then be applied to B. *K. A. Hirsch.*

Nowacki, Werner. Über die Anzahl verschiedener Raumgruppen. Schweiz. Mineral. Petrog. Mitt. 34, 160–168 (1954).

The author enumerates the abstractly distinct crystallographic groups. For point groups he finds, in one dimension, two: C_1 and C_2; in two dimensions, these and seven others:

$$D_2\simeq C_2\times C_2,\ C_4,\ D_4,\ C_3,\ D_3,\quad C_6\simeq C_2\times C_3,\quad D_6\simeq C_2\times D_3;$$

in three dimensions, these and nine others:

$$C_2\times C_2\times C_2,\quad C_2\times C_4,\quad C_2\times D_4,\quad C_2\times C_2\times C_3,\\ C_2\times D_6,\quad T,\quad C_2\times T,\quad O,\quad C_2\times O.$$

For space groups with finite fundamental regions, he finds that the seventeen two-dimensional groups of Pólya and Niggli [Z. Kristallographie 60, 278–298 (1924)] are all abstractly distinct. So likewise are the 219 three-dimensional groups of Fedorov, Schoenflies and Barlow. (The number 230, usually given in this connection, includes eleven enantiomorphous pairs; but of course each pair contributes only one to the list of abstract groups.) *H. S. M. Coxeter.*

Wever, Franz. Über die Kennzeichnung von Relationen endlicher Gruppen. Arch. Math. 5, 326–331 (1954).

A finite group G presented by k generators is the factor group F_k/R_i of the free group F_k modulo a relation group R_i. The intersection $W=\bigcap_i R_i$ is a characteristic subgroup of F_k contained in R_i. Also let T be the largest fully invariant subgroup of F_k contained in R_i. The "U group" $U=F_k/W$ and the "V group" $V=F_k/T$ are groups associated with G in the following way: U is generated by elements $x_1, \cdots, x_k$ satisfying relations $f(x_1, \cdots, x_k)=1$ which are satisfied by any set of k elements generating G. Similarly, V satisfies the identical relations satisfied by any set of k elements in G. Both U and V can be expressed as a subdirect product of a suficient number of replications of G: U, by taking k elements whose components are all possible sets of generators and generating from these a subgroup of the direct product; V similarly, by taking k elements in all possible ways. Associated with the symmetric group on three letters is the V group of order $2^2\cdot 3^8$ with relations $x^6=1$, $(x^2, y^2)=1$ holding for all elements. For the U group the order is $2^2\cdot 3^3$ and the characteristic relation $u^2(u^2, v)=1$ holds. To get G itself from two generators a and b we need exactly one of the following relations $a^2=1$, $(ab)^2=1$, or $b^2=1$.

Marshall Hall, Jr. (Columbus, Ohio).

Wielandt, Helmut. Zum Satz von Sylow. Math. Z. 60, 407–408 (1954).

The author proves the following theorem: If a finite group G, of order g, contains a nilpotent subgroup H, whose order h is prime to its index g/h, and if M is an arbitrary subgroup of G whose order m divides h, then M is contained in a conjugate subgroup of H in G. The proof is by a multiple induction, on the number of distinct prime factors of H, on g, and on m, and makes use of the fact that a finite group whose subgroups are all nilpotent is soluble. Whether it is sufficient to assume that H is supersoluble (instead of nilpotent) remains an open question. Solubility of H is not enough, for the simple group of order 168 contains two non-conjugate subgroups of order 24. *K. A. Hirsch.*

Čunihin, S. A. On factorization of finite groups. Doklady Akad. Nauk SSSR (N.S.) 97, 977–980 (1954). (Russian)

Let N be a finite collection of integers greater than one, in which some integers may appear more than once. An equivalence relation in N is defined: $n_1 \in N$ is said to be connected by the set N to $n_2 \in N$ provided there exists a finite chain of elements of N, of which the first term is n_1 and the last n_2 and such that every two adjacent terms are not relatively prime integers. The product of all integers in an equivalence class is called the block of this class of numbers in N. The blocks for the set N of all indices of a composition series of a finite group $\mathfrak{G}$ are called the composition blocks of $\mathfrak{G}$. To every factorization of the order $g>1$ of the finite group $\mathfrak{G}$ in the form $g=\prod_{j=1}^k h_j$, where each h_j is a composition block of $\mathfrak{G}$ or is a product of composition blocks of $\mathfrak{G}$, there exists a factorization $\mathfrak{G}=\mathfrak{H}_1\cdots\mathfrak{H}_k$, where each $\mathfrak{H}_j$ is a subgroup of $\mathfrak{G}$ with order h_j. In particular, if each h_j is a composition block, then the subgroups $\mathfrak{H}_1, \cdots, \mathfrak{H}_k$ may be chosen as pairwise permutable. Let h be a composition

block or a product of composition blocks of $\mathfrak{G}$. Every solvable subgroup of $\mathfrak{G}$ with order dividing h is contained in some subgroup with order h. If $\mathfrak{G}$ contains a solvable subgroup $\mathfrak{H}$ with order h, then every subgroup with order h is solvable and is conjugate to $\mathfrak{H}$. *R. A. Good.*

Huppert, Bertram. Normalteiler und maximale Untergruppen endlicher Gruppen. Math. Z. **60**, 409–434 (1954).

The paper contains a large number of results on the structure of finite soluble groups whose principal factors (orders of the factor groups of a principal series) are subjected to arithmetical restrictions. Only some of these results will be reported here. Apart from the principal series $G=N_0>N_1>\cdots>N_k=E$, where N_{i-1}/N_i is a minimal normal subgroup of G/N_i, the author considers "maximal chains" $G=U_0>U_1>\cdots>U_l=E$, in which U_{i+1} is a maximal subgroup of U_i. Let p be a prime divisor of the order of G. The exponent of the highest power of p among the principal factors of G is called the principal p-rank $r_p(G)$ of G, and the greatest p-rank for the various prime divisors of the order of G the principal rank or briefly the rank $r(G)$ of G. Groups for which $r(G)=1$ are supersoluble. $r_p(G)$ also turns out to be the exponent of the highest power of p that occurs in any maximal chain of G. If M, a maximal subgroup of a finite group G, is nilpotent and the Sylow subgroups of M are all regular (in the sense of P. Hall), then G is soluble. A finite group G also turns out to be soluble if every maximal subgroup of G is of prime index. In this case the derived group G' is proved to be nilpotent, and G even supersoluble. If the derived group G' of a finite group G is nilpotent and H is a subgroup of G whose order is prime to its index, then H' is a normal subgroup of G. In a finite group G with a nilpotent derived group G' the p-Sylow subgroup of the Frattini subgroup $\Phi(G)$ is the intersection of the Frattini subgroups of all the p-Sylow subgroups of G. If $G/\Phi(G)$ is supersoluble, then so is G. But it is not true that $r(G/\Phi(G))=r(G)$, as the author shows by a counterexample. If the indices in all maximal chains of G are primes or squares of primes, then G is soluble and $r(G)\leqq 2$. If in a soluble group G the indices of all maximal subgroups are primes or squares of primes, then the fourth derived group $G^{(4)}$ is nilpotent, and, when the order of G is odd, also G''; if also cubes of primes, but no higher powers occur among the indices of the maximal subgroups, then $G^{(6)}$ or, if G is of odd order, $G^{(4)}$ is nilpotent. Taunt has proved [Proc. Cambridge Philos. Soc. **45**, 24–42 (1949); these Rev. **10**, 351] that in a soluble group of cube-free order the third derived group is E. This is generalized as follows: If G is a soluble group whose Sylow subgroups are all abelian with at most k generators, then $G^{(3)}=E$ for $k\leqq 2$, and $G^{(4)}=E$ for $3\leqq k\leqq 5$. If all the minimal normal subgroups of a soluble group G with abelian Sylow subgroups have at most k generators, then $r(G)\leqq k$. The same conclusion holds if the indices of the maximal subgroups of G are prime powers with exponents at most k. It is well known that finite groups whose proper subgroups are all nilpotent are soluble, and if the order is divisible by at least three distinct primes, even nilpotent. The author proves a similar result in the supersoluble case: If every proper subgroup of the finite group G is supersoluble, then G is soluble, and if the order of G is divisible by at least four distinct primes, then G is supersoluble. If every maximal subgroup of a maximal subgroup of G is normal in G, then G is supersoluble, and if the order of G is divisible by at least three distinct primes, then G is nilpotent. If even the third term in every maximal chain of G is normal in G, then G is supersoluble, provided that the order of G is divisible by at least three distinct primes; also the derived group G' of G is nilpotent, and $r(G)\leqq 2$.

The proofs of all these and many other results contained in the paper make use of a surprisingly wide variety of methods: classical induction arguments, the finer Sylow structure, monomial and modular representation theory, a knowledge of the soluble subgroups of the projective unimodular groups $PSL(2, p)$ and $PSL(3, p)$ and so on.

K. A. Hirsch (Boulder, Colo.).

Huppert, Bertram, und Itô, Noboru. Über die Auflösbarkeit faktorisierbarer Gruppen. II. Math. Z. **61**, 94–99 (1954).

The first named author has proved in a recent paper [Math. Z. **59**, 1–7 (1953); these Rev. **15**, 197] that the product of a dihedral group with an abelian group or with a p-group is soluble. These results are generalized in the present note to the following theorem: The product of a dihedral group with a nilpotent group is soluble. The first factor need not even by dihedral; the existence of a cyclic subgroup of index 2 is sufficient. The full statement reads then: If $G=LN$, where L has a cyclic subgroup of index 2 and N is nilpotent, then G is soluble. The authors also prove the following "unsymmetrical" result: The group $G=UZ$ is soluble if Z is a cyclic group of odd order and U satisfies the conditions: U has a normal subgroup whose factor group is isomorphic to the 2-Sylow subgroups of U, and the derived group U' is nilpotent.

The first theorem is no longer true if L is subject only to the condition that it has a cyclic normal subgroup of prime index. This is shown by the factorisation of the simple group of order 168 into the product of a dihedral group of order 8 and a non-abelian group of order 21. In the second theorem Z cannot be replaced by a cyclic group of even order, and the prime number p occurring in the definition of U cannot be replaced by an odd prime number.

The proofs of the two theorems require a heavy apparatus, in particular, many of the results on factorisable groups that have been developed in recent years [see, for example, these Rev. **14**, 13]. *K. A. Hirsch* (Boulder, Colo.).

Barbilian, D. Sur les groupes sans torsion de A. I. Maltsev. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **3**, 475–497 (1952). (Romanian. Russian and French summaries)

The author searches for conditions under which the periodic elements of a group form a subgroup. This is known to be the case in all groups with the property that every proper subgroup is distinct from its normaliser, in particular, in all nilpotent and even locally nilpotent groups. [See, e.g., Kuroš, Teoriya grupp, 2nd ed., Gostehizdat, Moscow, 1953, p. 414; these Rev. **15**, 501.] As far as the reviewer has been able to ascertain, the two theorems of the present note refer to special cases of the above mentioned general result.

K. A. Hirsch (Boulder, Colo.).

Rédei, L., und Steinfeld, O. Gegenseitige Schreiersche Gruppenerweiterungen. Acta Sci. Math. Szeged **15**, 243–250 (1954).

Let A and B be two given groups. The authors search for all groups G that have a normal subgroup A', isomorphic to A, with $G/A'\cong B$, and in addition a normal subgroup B', isomorphic to B, with $G/B'\cong A$, in other words, that are extensions of A by B and at the same time extensions of B by A. One solution is, of course, the direct product of A

and B, and if A' and B' have a trivial intersection, it is the only solution. The groups G with the required properties are described by the authors in terms of four functions with two arguments each that are subject to fourteen conditions, and a further four functions of one argument that are subject to six conditions and have to satisfy very general permutational requirements. The treatment is based on the "skew" product of the first named author [J. Reine Angew. Math. **188**, 201–227 (1950); these Rev. **14**, 13]. *K. A. Hirsch.*

Halezov, E. A. Automorphisms of matrix subgroups. Doklady Akad. Nauk SSSR (N.S.) **96**, 245–248 (1954). (Russian)

Soit F_n le groupe de toutes les $n\times n$ matrices non singulières à éléments dans le corps F et soit F_n^r le demi groupe multiplicatif de toutes les $n\times n$ matrices de rang $\leqq r$ à éléments dans le corps F. Les équivalences sur F_n^r compatibles avec la multiplication ont été déterminées par Malcev [mêmes Doklady (N.S.) **90**, 333–335 (1953); ces Rev. **14**, 1057]. Les automorphismes de F_n pour $n\geqq 3$ sont connus [cf. R. Baer, Linear algebra and projective geometry, Academic Press, New York, 1952; ces Rev. **14**, 675]. L'auteur démontre par induction sur n que tout automorphisme θ de F_n^r est de la forme $\|x_{ij}\|^\theta=S\|x^\varphi{}_{ij}\|S^{-1}$ où $S\in F_n$ et φ est un automorphisme du corps F. Il resulte également de la démonstration que si les demi groupes F_n^r $(n\geqq 2)$ et G_m^s $(m\geqq 2)$ sont isomorphes alors $r=s$, $n=m$ et les corps F et G sont isomorphes. *J. Riguet* (Paris).

Bivins, Robert L., Metropolis, N., Stein, Paul R., and Wells, Mark B. Characters of the symmetric groups of degree 15 and 16. Math. Tables and Other Aids to Computation **8**, 212–216 (1954).

The authors used the electronic computer (MANIAC) to obtain tables of symmetric group characters of degrees 15 and 16. The mathematics of the method used was as follows.

If the class ρ' is obtained from the class ρ by the deletion of a single cycle of order m, then, for $(\lambda)=(\lambda_1, \lambda_2, \cdots, \lambda_k)$,

$$\chi_\rho^{(\lambda)}=\sum_\mu \chi_{\rho'}^{(\mu)},$$

where (μ) is summed for the partitions

$$(\lambda_1-m, \lambda_2, \cdots, \lambda_k),\quad (\lambda_1, \lambda_2-m, \cdots, \lambda_k), \cdots, (\lambda_1, \lambda_2, \cdots, \lambda_k-m).$$

If the parts in these partitions are not in descending order the parts are modified according to Murnaghan's conventions.

The two tables of characters have been photographed on microfilm and are available in the UMT File [UMT 195]. The computing time for $n=15$ was five hours; for $n=16$ twelve hours were necessary to compute 27,258 distinct characters. *D. E. Littlewood* (Bangor).

Burrow, M. D. A generalization of the Young diagram. Canadian J. Math. **6**, 498–508 (1954).

Young's method of obtaining primitive idempotents and irreducible representations of the symmetric group algebra is generalized as follows. Let two subgroups R and C of a group G have representations of the first degree, θ and φ respectively. If for any element $s\in G$ the condition $s\in CR\rightleftarrows\theta(r)=\varphi(c)$ holds for every pair of elements $r\in R$, $c\in C$ for which $srs^{-1}=c$, then $e=PN$ is a multiple of a primitive idempotent, where $P=\sum_{r\in R} r\theta(r)$, $N=\sum_{c\in C} c\varphi(c)$.

The method is applied to the group $\mathrm{GL}(2, q)$. The two subgroups considered are

$$R=\left\{\begin{pmatrix}\alpha & \beta\\ & 1\end{pmatrix}\right\}$$

of order $q(q-1)$ and a cyclic group C of order q^2-1 generated by an element of $\mathrm{GL}(2, q)$ similar to

$$\begin{pmatrix}\sigma & \\ & \sigma^q\end{pmatrix}$$

in which σ is a primitive root of the quadratic extension field $\mathrm{GF}(q^2)$.

For this group there are (a) $q-1$ irreducible representations of degree 1, (b) $q-1$ irreducible representations of degree q, (c) $\frac{1}{2}(q-1)(q-2)$ irreducible representations of degree $q+1$, (d) $\frac{1}{2}q(q-1)$ irreducible representations of $q-1$. The method gives the representations in each of the cases (a), (b), (c), but the author has failed to obtain results for the case (d). *D. E. Littlewood* (Bangor).

Robinson, G. de B. On the modular representations of the symmetric group. IV. Canadian J. Math. **6**, 486–497 (1954).

[For parts I–III see Proc. Nat. Acad. Sci. U. S. A. **37**, 694–696 (1951); **38**, 129–133, 424–426 (1952); these Rev. **13**, 530; **14**, 243, 244.] The representations of the symmetric group S_n correspond to partitions $(\lambda)=(\lambda_1, \lambda_2, \cdots, \lambda_r)$ of n, $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n$. With the partition is associated a diagram with λ_i nodes in the ith row. The graph $G[\lambda]$ is obtained by replacing the number $i-j$ for the node in the ith row, jth column. The q-graph is obtained by replacing $g_{ij}=i-j$ by g_{ij}, its least positive residue (mod q). The q-graph is denoted by $\mathrm{G}[\lambda]$.

For modular representation theory the q-core of a partition is important. If all possible kq-hooks are removed from the diagram of a partition (λ), then the diagram of a smaller partition, the q-core, is obtained. The weight of the partition (λ) is the number of q-hooks which can be removed to obtain the q-core.

If a single node is added to the diagram of a partition (λ), a diagram corresponding to a representation of S_{n+1} is obtained. If the process is restricted so that the added node corresponds to the residue r in the q-graph, the process is called r-inducing. The deletion of a node corresponding to a residue r in the q-graph is r-restricting. The author studies the effect of r-inducing and r-restricting on the weight of the partition. The number d of nodes of class r, i.e., corresponding to the residue r in the q-graph, which can be added to the graph is called the r-defect of $[\lambda]$. The number d^* of nodes of class r which can be removed is called the r-affect of the partition.

The author proves that the increase in weight of a partition which results from the addition of an r-node is $d-d^*-1$, and from the deletion of an r-node, d^*-d-1. The r-defect and the r-affect of the q-core of a partition (λ) are denoted respectively by δ and δ^*. If the r-defect of a q-core is δ, then the addition of δ r-nodes yields another q-core $[\lambda']$. If the r-affect of a q-core is δ^* the removal of δ^* r-nodes yields another q-core $[\lambda'']$.

The totality of diagrams obtained from a given diagram by r-inducing and r-restricting at all stages constitute a Boolean algebra. Write $[\lambda]=[\lambda^{d^*}]$ and put $d+d^*=l$, which is constant for all diagrams. The dimension of any diagram

of the set is defined to be d^*. The number of diagrams of dimension d^* is the binomial coefficient $\binom{l}{d^*}$. The weight of such a partition is given by $w=w_t+d^*(l-d^*)$.

The r-defect and the r-affect of the q-core are

$$\delta=\tfrac{1}{2}\{d-d^*+|d-d^*|\},\quad \delta^*=\tfrac{1}{2}\{d^*-d+|d-d^*|\}.$$

D. E. Littlewood (Bangor).

Harish-Chandra. Representations of semisimple Lie groups. V. Proc. Nat. Acad. Sci. U. S. A. 40, 1076–1077 (1954).

[For parts I–IV see these Rev. 13, 106, 107.] Let G be a semisimple Lie group. In terms of a (real) Cartan subgroup and a fixed ordering of the roots the author defines a certain (solvable) subgroup S. Lemma 1 asserts that the number of double cosets of S in G is finite. This had previously been announced by F. Bruhat [C. R. Acad. Sci. Paris **238**, 550–553 (1954); these Rev. **15**, 505] for the classical groups. For certain irreducible unitary representations of S the author considers the corresponding induced representation of G. Theorem 1 asserts that this is a quasi-simple unitary representation of G (in the sense of the author) and the multiplicity of each irreducible representation of K is finite. Here K denotes the inverse image in G of a maximal compact subgroup of the adjoint group G^*. The author considers representations π whose matrix coefficients are square-integrable functions on G^* and states that analogues of the orthogonality relation hold for them. This generalizes the discrete series of irreducible unitary representations of the 3-dimensional Lorentz group discovered by Bargmann [Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133], except that the question of irreducibility is left open.

F. I. Mautner (Princeton, N. J.).

Harish-Chandra. Representations of semisimple Lie groups. VI. Proc. Nat. Acad. Sci. U. S. A. 40, 1078–1080 (1954).

This is a continuation of the note reviewed above. The relation of the representations π introduced above with representations of the associative enveloping algebra of the Lie algebra is discussed, in particular, the problem of the existence of a highest weight. Finally the author introduces a Hilbert space of certain square-integrable holomorphic functions to serve as representation space for π, generalizing Bargmann's [Ann. of Math. (2) **48**, 568–640 (1947); these Rev. **9**, 133] Hilbert space of holomorphic functions. No proofs are given. *F. I. Mautner* (Princeton, N. J.).

Dynkin, E. B. Corrections to the paper, "Homologies of compact Lie groups." Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 233 (1954). (Russian)

See Uspehi Matem. Nauk (N.S.) 8, no. 5(57), 73–120 (1952); these Rev. **15**, 601.

Gluškov, V. M. On a class of noncommutative locally bicompact groups. Doklady Akad. Nauk SSSR (N.S.) 96, 229–232 (1954). (Russian)

A group is locally nilpotent (l.n.) if every subgroup generated by a finite subset is nilpotent [A. Kuroš, Teoriya grupp, 2nd ed., Gostehizdat, Moscow, 1953; these Rev. **15**, 501]. The aim of this paper is to investigate the structure of the l.n., locally compact (l.c.) groups. Let G be such a group, K be its connected component of the identity, and B be the union of all compact subgroups of G. Then B is a closed subgroup of G, $N=KB$ is an open normal subgroup of G, G/N is discrete, l.n. without torsion, and N is isomorphic with a factor group $(L\times B)/C$, where L is some connected, simply connected nilpotent Lie group and C is a closed central subgroup of $L\times B$ such that $C\cap B=\{e\}$; moreover, G is a generalized Lie group [see A. Gleason, Duke Math. J. **18**, 85–104 (1951); these Rev. **12**, 589] and it is Lie if and only if B is Lie. On account of the above results, the study of all l.n., l.c. groups is to a large extent reduced to the study of those groups which are also periodical, i.e. such that $G=B$. If G is periodical, it has open compact subgroups, and if H is such a subgroup, (G, H) is a direct product [see Vilenkin, C. R. (Doklady) Acad. Sci. URSS (N.S.) **47**, 611–613 (1945); these Rev. **7**, 241] of topological p-groups. Finally, the author studies the case when G is generated by some compact subset; let us point out the fact that, in this case, G is nilpotent if and only if B (which is compact) is nilpotent. No proofs are given but a few interesting lemmas are stated. *J. L. Tits* (Brussels).

Matsushima, Yozô. Sur le prolongement d'un pseudogroupe d'isomorphismes locaux d'une variété différentiable. Nagoya Math. J. 7, 103–110 (1954).

Let M be a differentiable manifold. By a local isomorphism of M is meant an isomorphism of an open submanifold (possibly empty) of M with an open submanifold of M. A pseudogroup of M is defined to be a set $\mathfrak{G}$ of local isomorphisms of M such that: a) the restriction of an element of $\mathfrak{G}$ to an open subset of its domain of definition (which the author calls its source) belongs to $\mathfrak{G}$; b) if $f,g\in\mathfrak{G}$, then $f\circ g\in\mathfrak{G}$; c) if $f\in\mathfrak{G}$, then $f^{-1}\in\mathfrak{G}$. The author proposes to define the notions of isomorphism and homomorphism for pseudogroups by a method similar to the one used by E. Cartan and E. Vessiot in the theory of infinite Lie groups. By a generalized fiber variety of base M is meant a variety E, given together with a differentiable mapping π of E onto M such that the rank of π is everywhere equal to the dimension of M. A local isomorphism $\bar{f}$ of E is called a prolongation of a local isomorphism f of M if the following conditions are satisfied: we have $\pi\circ\bar{f}=f\circ\pi$, and the image under π of the source of $\bar{f}$ is exactly the source of f. A pseudogroup $\bar{\mathfrak{G}}$ on E is called a prolongation of a pseudogroup $\mathfrak{G}$ on M if the following conditions are satisfied: a) every element of $\bar{\mathfrak{G}}$ is a prolongation of some element of $\mathfrak{G}$; b) every element of $\mathfrak{G}$ has at least one prolongation in $\bar{\mathfrak{G}}$. This prolongation is called an isomorphic prolongation if the following further conditions are satisfied: c) each $f\in\mathfrak{G}$ has a maximal prolongation $\bar{f}$ in $\bar{\mathfrak{G}}$ (i.e. every prolongation of f in $\bar{\mathfrak{G}}$ is a restriction of $\bar{f}$); d) if $\bar{f}$, $\bar{g}$ are the maximal prolongations in $\bar{\mathfrak{G}}$ of elements f,g of $\mathfrak{G}$, then $\bar{f}\circ\bar{g}$ is a prolongation of $f\circ g$; and, if g is a restriction of f, then $\bar{g}$ is a restriction of $\bar{f}$; e) if an $f\in\mathfrak{G}$ is the identity mapping of some open submanifold of M, then its maximal prolongation is the identity mapping of some open submanifold of E.

This being said, let $\mathfrak{G}$ and $\mathfrak{G}'$ be pseudogroups of transformations of manifolds M and M'. Then $\mathfrak{G}$ and $\mathfrak{G}'$ are called isomorphic if they have a common isomorphic prolongation; and $\mathfrak{G}'$ is said to be homomorphic to $\mathfrak{G}$ if there is a common prolongation of $\mathfrak{G}$ and $\mathfrak{G}'$ which is an isomorphic prolongation of $\mathfrak{G}$.

The essential result of the present paper is that the notions of homomorphism and isomorphism defined in this manner are transitive. This is proved by showing that any two prolongations $\bar{\mathfrak{G}}$, $\bar{\mathfrak{G}}'$ of a pseudogroup $\mathfrak{G}$ have a common prolongation, which is an isomorphic prolongation of $\bar{\mathfrak{G}}$ if

$\tilde{\mathfrak{G}}'$ is an isomorphic prolongation of $\mathfrak{G}$. This in turn is accomplished by a generalization of the notion of the fibered product of two fibered spaces.

If G is a Lie group, then the author associates to G the pseudogroup $\mathfrak{G}(G)$ of all local isomorphisms of the variety of G which are restrictions of left translations of G. It then surprisingly turns out that, G_1 and G_2 being Lie groups, a necessary and sufficient condition for $\mathfrak{G}(G_2)$ to be homomorphic (resp., isomorphic) to $\mathfrak{G}(G_1)$ is for G_2 to be algebraically a homomorphic (resp., isomorphic) image of G_1.

C. Chevalley (New York, N. Y.).

Schwarz, Š. Maximal ideals and the structure of semigroups. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **3**, 17–39 (1953). (Slovak. Russian summary)

The contents of this paper are the same as those of two previous papers by the author [Čehoslovack. Mat. Ž. **3(78)**, 139–153, 365–383 (1953); these Rev. **15**, 850].

A. H. Clifford (Baltimore, Md.).

Tamura, Takayuki. Supplement to the paper "On compact one-idempotent semigroups." Kōdai Math. Sem. Rep. 1954, 96 (1954).

See same vol. 17–21 (1954); these Rev. **15**, 933.

NUMBER THEORY

Thébault, Victor. Sur des produits de nombres entiers consécutifs. Mathesis **63**, 254–261 (1954).

Elementary proofs are given of special cases of known results on the impossibility of the product of consecutive integers being a square or a small multiple of a square.

I. Niven (Eugene, Ore.).

Robinson, Raphael M. Mersenne and Fermat numbers. Proc. Amer. Math. Soc. **5**, 842–846 (1954).

In 1952–53 the SWAC carried out, as a backlog program, tests for the primality of 2^n-1 for all primes $n<2304$ for which no factor of 2^n-1 had been discovered. The present paper is an account of the results obtained. All these Mersenne numbers were found to be composite except for the seventeen values: $n=2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281$, for which 2^n-1 is a prime. The last five are, by far, the largest known primes.

No details are given of the author's very effective code for carrying out the well known Lucas test. By a slight modification, the code becomes a test for the primality of Fermat numbers $2^{2^n}+1$. These tests were run for $n=7, 8, 10$. The results for $n=7, 8$ agreed with those obtained by Morehead [Bull. Amer. Math. Soc. **11**, 543–545 (1905)] and Morehead and Western [ibid. **16**, 1–6 (1909)]. The case $n=10$ tested composite also. Later the factor 45592577 of $2^{2^{10}}+1$ was discovered by Selfridge [Math. Tables and Other Aids to Computation **7**, 274–275 (1953)].

It is difficult, especially for those who have performed Lucas' test by desk calculator, to appreciate the speed with which the SWAC carries out this calculation. The time required is approximately $n^3/4$ microseconds.

Reference is made also to the fact that $2^{2191}-1$ is composite according to calculation made on the ILLIAC.

D. H. Lehmer (Berkeley, Calif.).

Bieberbach, Ludwig. Mathematische Fragen im Bereich der magischen Quadrate. Math.-Phys. Semesterber. **4**, 59–81 (1954).

A presentation of certain known facts about magic squares, mainly concerning the existence (and number, if known) of squares of certain special types. Half the paper is a condensation of the work of Rosser and Walker on pandiagonal squares [Duke Math. J. **5**, 705–728 (1939); these Rev. **1**, 133]. The rest is a summary of results obtained by the author and others on symmetric squares, Stifel squares, latin squares, cyclic squares, etc.

R. J. Walker (Los Angeles, Calif.).

Kale, M. N. A note on the magic square of 9 cells. Math. Student **22**, 144–145 (1954).

A formula for the number of such squares with positive integer elements and given central element.

R. J. Walker (Los Angeles, Calif.).

Alder, H. L. Note concerning a method for finding primes. Amer. Math. Monthly **61**, 698 (1954).

***Gelfond, Alexander O. Ganzzahlige Lösungen von Gleichungen.** Verlag von R. Oldenbourg, München, 1954. 59 pp. DM 7.80. Have

***Gelfond, A. O. Die Auflösung von Gleichungen in ganzen Zahlen (Diophantische Gleichungen).** Deutscher Verlag der Wissenschaften, Berlin, 1954. 59 pp. on order

Translations of the author's Rešenie uravnenii v celyh čislah [Gostehizdat, Moscow, 1952; these Rev. **13**, 913].

Utz, W. R. Diophantine equations. Pi Mu Epsilon J. **2**, 2–10 (1954).

Rivier, William. Sur les solutions entières et non négatives de l'équation $rx+sy=m$. Bull. Sci. Math. (2) **78**, 147–155 (1954).

Let $\omega(m)$ denote the number of non-negative integral solutions of $rx+sy=m$. The writer continus his study [same Bull. (2) **77**, 51–55 (1953); these Rev. **14**, 1063] of properties of $\omega(m)$, proving such theorems as the following. Let $k>0$, m_1 and m_2 be any non-negative integers satisfying $m_1+m_2=krs-r-s$; assume that the integers $r>0$, $s>0$ satisfy $(r, s)=1$, and that $m\geqq 0$. Then $\omega(m_1)+\omega(m_2)=k$.

I. Niven (Eugene, Ore.).

Sansone, G. Soluzioni intere delle equazioni $3y^4-2x^4=z^2$, $\lambda^4-6g^4=v^2$. Formule di Pepin e loro inversione. Matematiche, Catania **8**, no. 2, 3–10 (1953).

The infinitude of solutions in integers of $3y^4-2x^4=z^2$ is obtained by a geometric procedure. This contrasts with the recursion method of solution by T. Pepin [Atti Accad. Pont. Nuovi Lincei **31**, 397–427 (1878)]. Pepin's procedure is outlined in the present paper, and then transformed in order to obtain the infinitude of solutions of $\lambda^4-6g^4=v^2$, a related equation which was also solved by Pepin.

I. Niven (Eugene, Ore.).

Buquet, A. Etude des solutions rationnelles de l'équation diophantienne $G(x)=ax^4+bx^3+cx^2+dx+e=z^2$. Mathesis **63**, 240–250 (1954).

It has been known since Euler and Fermat that a series of rational solutions of the equation of the title can be ob-

tained from one such solution. The present paper continues the detailed study of various geometric properties of such chains of solutions, earlier papers having treated also the problem of a cubic polynomial equal to a square [Mathesis **59**, 233–236 (1950); **60**, 239–243 (1951); **61**, 183–193 (1952); **62**, 281–289 (1953); these Rev. **12**, 590; **13**, 535; **14**, 450; **15**, 400]. *I. Niven* (Eugene, Ore.).

James, R. D., and Niven, Ivan. Unique factorization in multiplicative systems. Proc. Amer. Math. Soc. **5**, 834–838 (1954).

Let M be a multiplicatively closed system of positive integers such that if $x \in M$ and $y \equiv x \pmod{n}$, $y > n$, then $y \in M$ and let n denote the smallest positive integer which can be used to define M. Further, suppose that A is the class of all numbers $1, 2, \cdots, n$ relatively prime to n, and B the class of all numbers $1, 2, \cdots, n$ not belonging to A. The following theorem is then proved. M has unique factorization if and only if $r \cap A = A$, $M \cap B = 0$.
H. Bergström (Göteborg).

Erdös, Paul. Some results on additive number theory. Proc. Amer. Math. Soc. **5**, 847–853 (1954).

If A is an increasing sequence of positive integers, let $A(n)$ denote the number of $a \in A$ satisfying $a \leqq n$. Two sequences A and B are said to be complementary if every sufficiently large integer can be represented in the form $a+b$, with $a \in A$, $b \in B$. A recent result due to G. G. Lorentz [Proc. Amer. Math. Soc. **5**, 838–841 (1954); these Rev. **16**, 113] states that for each sequence A there exists a complementary sequence B such that

$$(*) \qquad B(n) < c_1 \sum_{k=1}^{n} \frac{\log A(k)}{A(k)}.$$

If A is taken as the sequence of primes this inequality implies the existence of a sequence B satisfying

$$(**) \qquad B(n) < c_2 (\log n)^3,$$

and having the property that every sufficiently large number can be written in the form $p+b$, $b \in B$. Making use of the theorem of A. E. Ingham on the difference between consecutive primes [Quart. J. Math., Oxford Ser. **8**, 255–266 (1937)], the author improves Lorentz's estimate (**) to $B(n) < c_3 (\log n)^2$.

Again, let A be a sequence with a positive lower density. It then follows from (*) that A possesses a complementary sequence B with $B(n) < c_4 (\log n)^2$. The author shows that this estimate is best possible by proving the following result: there exists a sequence A, with positive lower density, such that every complementary sequence B of A satisfies $B(n) > c_5 (\log n)^2$. A number of subsidiary problems are also considered. *L. Mirsky* (Sheffield).

Erdös, P. On a problem of Sidon in additive number theory. Acta Sci. Math. Szeged **15**, 255–259 (1954).

Let $0 < a_1 < a_2 \cdots$ be an infinite sequence of positive integers. Denote by $f(n)$ the number of solutions of $n = a_i + a_j$. About twenty years ago Sidon, in an oral communication to Erdös, raised the question whether there exists a sequence a_i satisfying $f(n) > 0$ for all $n > 1$ and $\lim f(n)/n^\epsilon = 0$ for all $\epsilon > 0$. In the present note Erdös constructs such a sequence. In fact his sequence satisfies

$$(*) \qquad 0 < f(n) < c_1 \log n$$

for all $n > 1$. (The c's denote suitable positive absolute constants.) The idea of the construction is the following. Define $A = A_k$ as the least integer greater than $c_2 k^{1/2} 2^{k/2}$. Pick in all possible ways A integers from the interval $(2^k, 2^{k+1})$. The number N of such ways is $\binom{q}{A}$, $q = 2^k$. One thus obtains the integers $b_1^{(k)} < b_2^{(k)} < \cdots < b_A^{(k)}$. It is shown that if for each k one neglects $o(N)$ "bad" choices of the $b_i^{(k)}$ and forms a sequence $a_1 < a_2 < \cdots$ from any of the "good" choices of the $b_i^{(k)}$ ($k = 1, 2, \cdots; 1 \leqq i \leqq A$), then the sequence $a_1 < a_2 < \cdots$ will satisfy (*). Thus, roughly speaking, (*) will be satisfied for almost all choices of the $b_i^{(k)}$. *A. L. Whiteman.*

Stewart, B. M. Sums of distinct divisors. Amer. J. Math. **76**, 779–785 (1954).

For a given positive integer M, let $\alpha(M)$ denote the number of integers n which can be written in the form $n = \sum d$, where the d are distinct positive divisors of M. Then obviously $\alpha(M) \leqq \sigma(M)$, where $\sigma(M)$ is the sum of all positive divisors of M. For a given set S of integers M, let S^* be defined as the subset of S consisting of those integers for which $\sigma(M) - \alpha(M)$ is minimal. The author obtains complete characterizations of the sets I^* and O^*, where I is the set of all positive integers and O is the set of all odd integers $\geqq 5$. He also shows that the values of the function $\alpha(M)/\sigma(M)$ are everywhere dense on the interval $(0, 1)$.
L. Mirsky (Sheffield).

Zulauf, Achim. Über die Darstellung natürlicher Zahlen als Summen von Primzahlen aus gegebenen Restklassen und Quadraten mit gegebenen Koeffizienten. II. Die singuläre Reihe. J. Reine Angew. Math. **193**, 39–53 (1954).

The author investigates the singular series which occurs in his Paper I of the same title [J. Reine Angew. Math. **192**, 210–229 (1953); these Rev. **15**, 778].
T. Estermann (London).

Zulauf, Achim. Über die Darstellung natürlicher Zahlen als Summen von Primzahlen aus gegebenen Restklassen und Quadraten mit gegebenen Koeffizienten. III. Resultate für "fast alle" Zahlen. J. Reine Angew. Math. **193**, 54–64 (1954).

The author considers representations of natural numbers in the forms $p_1 + p_2$ and $p_1 + b_1 g_1^2 + b_2 g_2^2$, where p_1 and p_2 are odd primes in given classes of residues, b_1 and b_2 are given natural numbers, and g_1 and g_2 are integers. He proves that almost all natural numbers which satisfy certain congruences, necessary for trivial reasons, have such representations. ["b_r" in the fifth line of the introduction is a misprint for "g_r".] *T. Estermann* (London).

Gel'fond, A. O. On the partition of the natural series into classes by the group of linear substitutions. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 297–306 (1954). (Russian)

Let $m \geqq 2$, $p_0 \geqq 1$, $p_1 \geqq 0, \cdots, p_{m-1} \geqq 0$ be integers, and consider the set of linear substitutions

$$L_k(x) = mx + p_k m + k \quad (0 \leqq k \leqq m-1).$$

Denote by $L_k^{-1}(x)$ the integer y, if it exists, such that $x = L_k(y)$. Two positive integers N and n are said to belong to the same class if there exists a relation of the type

$$L_{k_1}^{\epsilon_1} \cdots L_{k_s}^{\epsilon_s}(N) = L_{q_1}^{\delta_1} \cdots L_{q_r}^{\delta_r}(n),$$

where the ϵ's and δ's have the values ± 1, and the k's and q's belong to the range $[0, m-1]$. In this way the set of positive integers is partitioned into disjoint classes, and it is not difficult to verify that the number of these classes is finite.

Let ν be the least integer of some class, and denote by $N_\nu(x)$ the number of integers, not exceeding x, which belong to the same class as ν. The main object of the paper under review is to study the behaviour of $N_\nu(x)$ for $x\to\infty$. The author shows, in the first place, that

$$N_\nu(x)=x\phi_\nu\left(\frac{\log x}{\log m}\right)+O(1),$$

where $\phi_\nu(t)$ is a periodic function with period 1. Now, let $\nu=n_0<n_1<n_2<\cdots$ be the integers of the class of ν. By investigating the properties of the power series $f_\nu(x)=\sum_{k=0}^\infty x^{n_k}$ and making use of the methods of complex function theory the author obtains an explicit, though complicated, formula for $\phi_\nu(t)$. The paper concludes with a discussion of the zeta-function associated with the class of ν, i.e. the function $\zeta_\nu(s)=\sum_{k=0}^\infty n_k^{-s}$. *L. Mirsky* (Sheffield).

Mordell, L. J. On intervals containing an affinely equivalent set of n integers mod k. Proc. Amer. Math. Soc. **5**, 854–859 (1954).

Let $n>2$ be a given positive integer and $(x)=(x_1, \cdots, x_n)$ a given set of n integers. Let k be a given positive integer. Improving a result of L. Rédei [Acta Math. Acad. Sci. Hungar. **2**, 75–82 (1951); these Rev. **13**, 627], the author first proves Theorem I: Let δ be the greatest divisor of k such that $x_1\equiv x_2\equiv\cdots\equiv x_n \pmod{\delta}$, and so $\delta=(x_1-x_n, x_2-x_n, \cdots, x_{n-1}-x_n, k)$. Then integers $a\not\equiv 0 \pmod{k}$, b exist such that an integer set (y), defined by $y_r\equiv ax_r+b \pmod{k}$ $(r=1, 2, \cdots, n)$, lies in an interval of width $L=L(\delta, k)$, where $nL^{n-1}\leqq 2^{n-1}\delta k^{n-2}$. Making applications of Dirichlet's classical distribution result and a general principle in the geometry of numbers the author next proves two theorems of which the following is Theorem II: Let λ be the least positive integer which satisfies the inequality $(\lambda+1)^n-\lambda^n\geqq k^{n-2}$. Then we can take $L=2\lambda$ or $L=2\lambda+1$ according as the inequality or equality sign holds. Under certain conditions, as stated in Theorem III, the result $L=2\lambda+1$ is best possible. Further, the result $L=2\lambda$ can be replaced by $L=2\lambda-1$ if now λ is the least positive integer which satisfies the inequality

$$(\lambda+1)^n-(\lambda-1)^n\geqq 2k^{n-2}.$$

A. L. Whiteman (Los Angeles, Calif.).

Kanold, Hans-Joachim. Über die Dichten der Mengen der vollkommenen und der befreundeten Zahlen. Math. Z. **61**, 180–185 (1954).

Let $Q^2(a)$ denote the largest square factor of a positive integer a, so that $a=f(a)Q^2(a)$ where $f(a)$ is square-free, and let $V(a)$ be the number of prime factors of a. Let p_n be the nth prime factor of a, and c any positive constant. Let $d(A)$ denote the asymptotic density of a set A of integers $\{a\}$. If for every $a\in A$, $a>c$, we have $Q^2(a)\geqq g(a)$ where $g(x)$ is, for $x>c$, a monotonic function with $g(\infty)=\infty$, then $d(A)=0$. If for fixed positive $\epsilon<1$ every $a\in A$ has the property that the integer $Q(a)$ has a prime factor p_n with $n\geqq\epsilon V(a)$, then $d(A)=0$. The set of perfect numbers, i.e. numbers n such that the sum of the divisors $\sigma(n)=2n$, has density zero. Finally, the set A of all amicable numbers, i.e. numbers m_1, m_2 such that $\sigma(m_1)=\sigma(m_2)=m_1+m_2$, satisfies $d(A)<0.204$. *I. Niven* (Eugene, Ore.).

Hua, Loo-Keng. On the number of solutions of Tarry's problem. Acta Sci. Sinica **1**, 1–76 (1952).

Let $r_t(P, k)$ be the number of integral solutions of the system of equations

$$x_1^h+\cdots+x_t^h=y_1^h+\cdots+y_t^h,\quad h=1, \cdots, k,$$

subject to the restriction that all variables lie in the closed interval $[1, P]$. The main result is that if $k\geqq 2$ and $t>t_0(k)$, then for large P

$$r_t(P, k)=\vartheta_t(k)\mathfrak{S}_t(k)P^{2t-k(k+1)/2}+O(P^{2t-k(k+1)/2-c(k)})$$

for some $c(k)>0$ and for certain positive $\vartheta_t(k)$, $\mathfrak{S}_t(k)$, $t_0(k)$, formulas for which are given. Here $t_0(2)=3$ and $t_0(k)\sim 3k^2\log k$ for large k. This improves a result given in the first edition of the author's book (see the following review) where the result was proved for $t>t_1(k)\sim\frac{3}{2}k^3\log k$. The sharpening of the lower bound for t results from the use of an improved form of Vinogradov's mean-value theorem proved earlier by the author [Quart. J. Math., Oxford Ser. **20**, 48–61 (1949); these Rev. **10**, 597].

In connection with the singular series $\mathfrak{S}_t(k)$, whose terms are non-negative, the author proves that it converges for $t>\frac{1}{2}k(k+1)+1$ if $k\geqq 3$ and diverges for all other t. The proof of divergence is rather easy. For $t>k^2$, and even for $t>\frac{1}{2}k(k+1)$, the proof of convergence is not too difficult and depends on earlier results of the author which are now well known. But the complete result on convergence is based on a very detailed study of the singular series and ultimately depends on an analysis of the solutions of the polynomial congruence $g(x)\equiv 0 \pmod{p}$.

The factor $\vartheta_t(k)$ is a k-fold infinite integral with non-negative integrand. By using the Young-Hausdorff theorem regarding a function and its Fourier transform, the author is able to show that $\vartheta_t(k)$ converges for $t>\frac{1}{2}k(k+2)$.

A number of remarks and statements of results are made concerning the Prouhet problem and the number

$$r_t(N_1, \cdots, N_k; P)$$

of solutions of

$$(*)\qquad x_1^h+\cdots+x_t^h=N_h\quad (h=1, \cdots, k)$$

subject to $1\leqq x_j\leqq P$. Finally an asymptotic result is obtained for $r'(N)$ which is defined as the number of solutions of

$$x_1+x_2+x_3=y_1+y_2+y_3,\quad x_1^2+x_2^2+x_3^2=y_1^2+y_2^2+y_3^2\leqq N.$$

This result is

$$r'(N)=35(3/4)^{1/2}N^{3/2}\log N+O(N^{3/2}\log^{1/2}N).$$

The proof depends on the fact that $r'(N)$ is the sum of squares of terms $r_3(N_1, N_2; N)$. Each of these terms is closely connected with $\Psi(m)$, the number of solutions of $X_1^2+X_1X_2+X_2^2=m$. Finally, $\Psi(m)$ is six times the sum of the Jacobi symbol $(-3|l)$ taken for $l|m$. By using these results and careful estimations, the result for $r'(N)$ is obtained. *L. Schoenfeld* (Princeton, N. J.).

***Hua, Loo-Keng. Tui Lei Su Shu Lun. [Additive prime number theory.]** Chung Kuo Ko Hsueh Yuan [Academia Sinica], Peking, 1953. 206 pp.

This book is a revised translation of the original Russian edition previously reviewed [Trudy Mat. Inst. Steklov. **22** (1947); these Rev. **10**, 597]. Aside from some changes in detail, the principal differences are as follows. Chapter V contains the author's improved version of Vinogradov's mean-value theorem referred to in the preceding review. This improvement leads to sharper results throughout the book.

Thus, in Chap. VII it is shown that the asymptotic formula for the number of representations of an integer as the sum of s kth powers of primes is valid for $s\geqq s_0\sim 4k^2\log k$ in place of the earlier result $s\geqq s_1\sim k^3\log k$. In Chap. IX, the author proves that $H(k)\leqq s_2\sim 4k\log k$ in place of the former $H(k)\leqq s_3\sim 6k\log k$. Chap. X gives the improved

result on Tarry's problem quoted in the preceding review. For the number of solutions of (*) of that review where the unknowns x_j are now taken to be primes, the validity of the asymptotic formula is proved for $s \geq s_4 \sim 6k^2 \log k$ in place of the earlier $s \geq s_5 \sim 4.14k^2 \log k$.

Finally, improved results on the singular series are obtained. The addendum and the English summary have been omitted. *L. Schoenfeld* (Princeton, N. J.).

Pyateckiĭ-Šapiro, I. I. Abelian modular functions. Doklady Akad. Nauk SSSR (N.S.) **95**, 221–224 (1954). (Russian)

A discussion of the author's extension of Siegel's definitions of modular group and modular functions of degree (genus) n [Math. Ann. **116**, 617–657 (1939); these Rev. **1**, 203] to the following. Let Ω be the set of matrices ω with p rows and $2p$ columns $\omega_1, \cdots, \omega_{2p}$ for which there exists a non-degenerate Abelian function with periods $\omega_1, \cdots, \omega_{2p}$. Let $\mathfrak{A}$ be the multiplication algebra of some irreducible matrix ω. Denote by R_0 a rational skew-symmetric matrix of order $2p$, and let $\Omega_{\mathfrak{A}}$ be the set of all ω in Ω with multiplication algebra $\mathfrak{A}$ and such that $\omega R_0 \omega' = 0$ and $\bar{\omega} R_0 \omega' > 0$. Then the modular group $\Gamma_{\mathfrak{A}}$ on $\Omega_{\mathfrak{A}}$ is the set of all unimodular matrices U of order $2p$ such that if $\omega \in \Omega_{\mathfrak{A}}$, then also $\omega U' \in \Omega_{\mathfrak{A}}$. *W. H. Simons* (Vancouver, B. C.).

Vinogradov, I. M. Distribution according to a prime modulus of prime numbers with a given value of the Legendre symbol. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 105–112 (1954). (Russian)

Let $\psi(z)$ be a periodic function of z with period 1, defined to be $1-\sigma$ if $0 \leq z < \sigma$ and to be $-\sigma$ if $\sigma \leq z < 1$. Let q be a sufficiently large prime. The paper is concerned with the distribution of the primes $p \leq N$ (where N is sufficiently large) for which the Legendre symbol $(p|q)$ has a given value. The main result is that, for the sum extended over these primes,

$$\sum \psi(ap/q) = O(N^{\epsilon}(N^{5/6} + Nq^{-1/2} + N^{1/2}q^{1/2}))$$

for any fixed $\epsilon > 0$, where a denotes any integer $\not\equiv 0 \pmod q$. The proof is arithmetical (i.e. without the use of exponential sums), and the method is an extension of that used by the author in a recent paper [same Izvestiya **17**, 3–12 (1953); these Rev. **15**, 855], from which two lemmas are quoted. An important part in the proof is played by another lemma, similar in its general nature to Lemma 5 of Chapter 9 of the author's paper, Trudy Mat. Inst. Steklov. **23** (1947) [these Rev. **10**, 599; see these Rev. **15**, 941 for an English translation]. *H. Davenport* (London).

★Selberg, Sigmund. Über die Summe $\sum_{n \leq x} \frac{\mu(n)}{nd(n)}$. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 264–272 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author discusses the sum above, giving first some elementary estimates and then a precise asymptotic estimate obtained by means of classical analytic techniques. *R. Bellman* (Santa Monica, Calif.).

Val'fiš, A. Z. On Euler's function. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **19**, 1–31 (1953). (Russian)

This paper is a complete account of the derivation of the following result, reported in the author's note of the same title [Doklady Akad. Nauk SSSR (N.S.) **90**, 491–493 (1953); these Rev. **15**, 11]:

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2}x^2 + O(x(\log x)^{2/3}(\log\log x)^2),$$

where $\varphi(n)$ is Euler's φ-function. Briefly, the method is as follows. It is first shown that

$$(*) \qquad \sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2}x^2 - x \sum_{n \leq x} \frac{\mu(n)}{n} \psi\left(\frac{x}{n}\right) + Bx,$$

where $\mu(n)$ is the Möbius function and $\psi(v) = v - [v] - \frac{1}{2}$. $\psi(x/n)$ is expanded in a Fourier series and improved estimates for the trigonometric sums arising from the right side of (*) are obtained using Vinogradoff techniques. The bulk of the paper is devoted to obtaining these estimates. *W. H. Simons* (Vancouver, B. C.).

Nöbauer, Wilfried. Über eine Gruppe der Zahlentheorie. Monatsh. Math. **58**, 181–192 (1954).

Let A_n denote the ideal of residual polynomials (mod n), let G_n be the group of the classes of polynomials (mod A) that possess inverses, and let P_n be the (abelian) subgroup of G_n containing polynomials of the form x^k. The present paper is concerned with the properties of P_n and makes use of the results of an earlier paper by the same author [Österreich. Akad. Wiss. Math. Nat. Kl. S.-B. IIa. **162**, 207–233 (1953); these Rev. **15**, 856]. The main results of the paper are the following. I. If n is not quadratfrei the group P_n is of order 1. If $n = p_1 p_2 \cdots p_r$, where the p_i are distinct primes, then P_n is isomorphic to the multiplicative group of a reduced residue system (mod $[p_1-1, \cdots, p_r-1]$). II. If n is quadratfrei and $n = a_1 a_2 \cdots a_t$, then P_n is a subgroup of the direct product $D = \prod P_{a_i}$ and the factor group $D/P_n = \prod_1^{t-1} B_j$, where B_j is the multiplicative group (mod b_j) and the b_j are determined by the a_i. III. If n is quadratfrei and m/n then P_m is a homomorphic image of P_n. IV. Condition that P_n be cyclic. V. Condition that $P_n \sim \prod P_{a_i}$ (compare II). VI. Given r distinct polynomials of the form x^k, there exist only a finite number of values of n for which the $\{x^k\}$ represent the group P_n. The paper closes with some numerical data. *L. Carlitz.*

★Kolden, K. On the prime divisors of homogenous polynomials which decompose into linear factors in a normal field. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 139–159 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The paper is concerned with a characterization of the (rational) prime division of homogeneous polynomials that factor completely in a normal field. Let $F(\theta)$ be a normal field of degree n with group G. Consider first the binary case

$$f(x, y) = \prod_{s \in G} (\alpha^{(s)}x + \beta^{(s)}y) = Ax^n + By^n + \cdots,$$

where $\alpha, \beta \in F(\theta)$; $A, B, \cdots$ are rational integers. If p is a prime divisor of $f(x, y)$ and $p = \mathfrak{p}\mathfrak{p}' \cdots \mathfrak{p}^{(n-1)}$ is the prime-ideal factorization of p in $F(\theta)$, then p is called a prime of first degree in $F(\theta)$; otherwise p is called exceptional. It is proved that if $\Delta(f(x, y))$, the discriminant of $f(x, y)$, does not vanish and p is a prime divisor of $f(x, y)$ such that $p \nmid (x, y)AB$, then either p is of first degree in $F(\theta)$ or is a factor of $\Delta(f)$. If $\Delta(f) = 0$, then $f(x, y) = (f_1(x, y))^m$, where $f_1(x, y)$ factors completely in some subfield of $F(\theta)$. In this case, all prime divisors of $f(x, y)$ are exceptional in $F(\theta)$;

if however the subfield is normal the prime divisors will be of first degree in the subfield with at most a finite number of exceptions. Turning next to the ternary case

$$f(x, y, z)=\prod_{s\in G}(\alpha^{(s)}x+\beta^{(s)}y+\gamma^{(s)}z)=Ax^n+By^n+Cz^n+\cdots,$$

the author defines the discriminant $\Delta(f(x, y, z))$ and also certain binary elements. It is proved that if p is a prime divisor of $f(x, y, z)$ such that $p\nmid(x, y, z)$, $p\nmid ABC$, then p is either of first degree in $F(\theta)$ or is a factor of $\Delta(f)$. If $\Delta(f)=0$, then $f(x, y, z)$ or one of its eliminants is equal to a power of a form which is completely reducible in a subfield of $F(\theta)$. In this case $f(x, y, z)$ may have infinitely many exceptional prime divisors where those that are not factors of the discriminant of $F(\theta)$ will occur to a power p^e, $(e, n)>1$. The above results are now extended to the general case of N indeterminants. It is observed that the case $N\geqq n$ is of no interest since in this event the given form is divisible by prime ideals of all kinds in $F(\theta)$. Assuming $N<n$ it is necessary to define the discriminant of a factorable form $f(x_1, \cdots, x_N)$ as well as the system of eliminants. The statement of the main theorem is similar to that in the ternary case. The writer considers a few special cases. Thus let

$$(*)\qquad f(x, y, z)=\prod_{(r,n)=1}(x+\epsilon^r y+\epsilon^{2r}z)\quad(\epsilon=e^{2\pi i/n}),$$

so that the group G is of order $\phi(n)$. Since G contains elements of order 2 it follows that $\Delta(f)=0$ and therefore $f(x, y, z)$ admits of infinitely many exceptional prime divisors. The paper closes with some numerical data concerning (*) in the cases $n=5, 7$.
L. Carlitz.

Urazbaev, B. On the density of the integral points of completely critical cyclic fields of degree p^m. Izvestiya Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. 1(6), 115–122 (1952). (Russian. Kazak summary)

A completely critical cyclic field of degree $n=p^m$ (p prime) is one in which each rational prime dividing the discriminant becomes the nth power of an ideal. Such fields have been considered by the author in a paper unavailable to the reviewer [On the discriminant of completely critical fields, Učenye Zapiski Kazah. Gos. Univ. **1951**]. In particular, it is stated without proof that all such fields have discriminants of the form $D=D_1=d_1^{n-1}$ or $D=D_2=d_2^{n-1}$, where d_1, d_2 are respectively of the type $d_1=q_1\cdots q_k$, $d_2=pq_1\cdots q_k$ and the q_j are distinct primes with $q_j=1$ (mod n). The number of distinct such fields for given D_1, D_2 is stated to be

$$\frac{a^k-b^k}{a-b},\quad\frac{a^{k+1}-b^{k+1}}{a-b}\quad(a=p^m-1,\ b=p^{m-1}-1)$$

respectively, but the reviewer cannot confirm this. Let $M_n(R)$ be the number of points in n-dimensional space whose euclidean orthogonal co-ordinates are the conjugate values of some integer in some such field of degree n. The author purports to prove that

$$\lambda_1R^n+O(R^{n-1})\leqq M_n(R)\leqq\lambda_2R^n+O(R^{n-1})$$

for some positive constants λ_1, λ_2. The proof leans heavily on the paper reviewed below and on a paper not available to the reviewer [Izvestiya Akad. Nauk Kazah. SSR **1951**, no. 62, Ser. Mat. Meh. **5**, 25–36 (1951); these Rev. **15**, 403]. For any one field the number of points in the sphere of radius R is (manifestly)

$$V_0R^nD^{-1/2}+O(R^{n-1}S_0\delta D^{-1/2}),$$

where V_0, S_0 are the volume and area of surface of the unit sphere, D is the discriminant and δ the diameter of a fundamental parallelepiped of the lattice of integer points. The proof then depends on a crude estimate for δ and a summation over all fields of given n. It appears to the reviewer that the analysis is inadequate, e.g., no account is taken of points in subfields which may be counted multiply.
J. W. S. Cassels (Cambridge, England).

Urazbaev, B. Asymptotic estimate of the density of completely critical cyclic fields of degree p^m. Izvestiya Akad. Nauk Kazah. SSR **1952**, no. 116, Ser. Astr. Fiz. Mat. Meh. 1(6), 123–133 (1952). (Russian. Kazak summary)

Here the author estimates the number $L(N)$ of completely critical cyclic fields of degree $n=p^m$ and of discriminant less than N^{n-1} [see the preceding review]. He obtains

$$\kappa N+O(N^{1-1/\nu})\leqq L(N)\leqq\kappa' N^2+O(N^{1-1/\nu}),$$

where $\nu=p^{m-1}(p-1)$, and indicates a proof of

$$L(N)<\kappa_1(\epsilon)N^{1+\epsilon}+O(N^{1+\epsilon-1/\nu})$$

(sic) where κ, κ', $\kappa_1(\epsilon)$ are positive constants and $\epsilon>0$ is arbitrary. He uses a relation between the number of completely critical fields of discriminant $D=d^{n-1}$ and the number of ideals of norm d in the field of nth roots of unity.
J. W. S. Cassels (Cambridge, England).

Hasse, Helmut. Artinsche Führer, Artinsche L-Funktionen und Gaussche Summen über endlich-algebraischen Zahlkörpern. Acta Salmanticensia. Ciencias: Sec. Mat. no. 4, viii+113 pp. (1954).

This memoir contains explicit proofs for some of the theorems and conjectures stated by the author in a previous paper [Abh. Deutsch. Akad. Wiss. Berlin. Kl. Math. Nat. **1952**, no. 1; these Rev. **15**, 404]. As stated (loc. cit.), the author's goal is to find arithmetic characterizations of generalized Gaussian sums belonging to arbitrary normal extensions N of a finite algebraic number field K. The methods of this study are fashioned in analogy to those which were successful for abelian extensions. In the latter case, class field theory permits on one side a representation of the Gaussian sum, initially defined by means of the functional equation for an L-series, as a sum involving congruence characters of K and the exponential function, and on the other side the representation of the given global Gaussian sum as a product of Gaussian sums in the p-adic completions of K. In this connection the author shows in every detail how his Gaussian sums can be expressed in terms of idèles and how essential simplifications may thus be achieved. Using results of Lamprecht [Math. Nachr. **9**, 149–196 (1953); these Rev. **14**, 942], the previously announced reduction of Gaussian sums in p-adic fields to Gaussian sums [with parameter, see the author's paper Abh. Deutsch. Akad. Wiss. Berlin. Math.-Nat. Kl. **1951**, no. 1; these Rev. **13**, 324] in finite fields is carried out in sufficient detail for odd primes, and for $p=2$ the reduction to the zeros of L-series for certain cyclic function fields of degree 4 is described [see H. L. Schmid, Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1941**, no. 14 (1942); these Rev. **8**, 310]. Thus multiplicative expressions for abelian Gaussian sums are obtained. Suppose now that ψ is a simple character of the sub-group H of the abelian Galois group G of N/K which induces the character ψ^* of G, then $\psi^*=\sum\chi\psi'$, where χ runs over the simple characters of G/H and ψ' is a prolongation of ψ to G. Corresponding to this relation, there

are multiplicative relations between the associated Gaussian sums which follow the conductor-discriminant formula of class-field theory and product relations of L-series. The author poses the problem to prove these multiplicative relations with arithmetic means based upon the arithmetic definition of the Gaussian sums. Again the formalism of Lamprecht (loc. cit.) and the interpretation of Gaussian sums for finite fields as factors of the functional equations for L-series belonging to function fields, and as zeros of such L-series, serve as an important guide to develop a plan of procedure [see H. Davenport and H. Hasse, J. Reine Angew. Math. **172**, 151–182 (1934); also A. Weil, Trans. Amer. Math. Soc. **73**, 487–495 (1952); these Rev. **14**, 452].

It is shown that the analytically proved relations which lead to product relations for the Gaussian sums relative to the corresponding local characters can be established arithmetically for abelian extensions without higher ramification by means of the reduction to Gaussian sums over finite fields. As a consequence the global product formulas $\prod \tau_{\mathfrak{k}}(\chi\psi')/\tau_{\mathfrak{k}}(\chi) = \tau_\Lambda(\psi)$, where Λ is the fixed field of H and the simple characters χ are taken mod H, is obtained by multiplying together all local product formulas $[\tau_{\mathfrak{k}}(\chi) = \sum \chi(\mathbf{x})\mathfrak{k}(x)$; see these Rev. **15**, 404]. As far as the method is concerned, the author's proof is quite straightforward, once the basic idea is grasped, but it is technically quite involved. Nevertheless it appears to be indicated in view of Lamprecht's results (loc. cit.) that the general abelian case can be settled in a similar manner. The remainder of the paper is devoted (i) to a new, somewhat simplified, presentation of Artin's theory of conductors and L-series for non-abelian extensions, which makes use of R. Brauer's reduction of non-abelian characters to sums of induced abelian characters, and J. Herbrand's theory of higher ramification groups; (ii) to the non-invariant definition of the local components $\tau_{\mathfrak{p}}$ of non-abelian Gaussian sums [which undoubtedly was motivated by the successful use of Brauer's results in (i); see these Rev. **15**, 404]; and (iii) to proofs of the more formal properties concerning the absolute values and nature of the Gaussian sums as elements in cyclotomic fields. Although the generalized local characters $\tau_{\mathfrak{p}}$ are not invariantly defined, they satisfy a parameter rule $\tau_{\mathfrak{p}}(\chi, N/K)^a = |\chi|_{\mathfrak{p}}(a_p)$, where $a = 1-(a_p)$, a_p a p-adic unit and $|\chi|_{\mathfrak{p}}$ the $\mathfrak{p}$-component $|\chi|(g) = \det D_\chi(g)$, $g \in G$, with the representation D_χ belonging to the character χ of G. The proof of this rule employs an interesting application of the theory of the norm-residue symbol of the field of all p^nth roots of unity. *O. F. G. Schilling* (Chicago, Ill.).

Chalk, J. H. H. On the primitive lattice points in the region $|(x+c)y| \leqslant 1$. Quart. J. Math., Oxford Ser. (2) **5**, 203–211 (1954).

Let $L_1 = \alpha x + \beta y$, $L_2 = \gamma x + \delta y$ be real forms with

$$\Delta = \alpha\delta - \beta\gamma \neq 0$$

and $c \neq 0$. Then there exist coprime integers x, y such that $|(L_1+c)L_2| \leqq (1-2^{-1/2})|\Delta|$, with equality necessary if and only if $L_1 = \lambda_1 y$, $L_2 = \lambda_2(x+2^{1/2}-1)y$, $c = -(1-2^{-1/2})\lambda_1$, apart from an integral unimodular substitution on x, y. Furthermore, this value of c is best possible even if α/β is assumed to be irrational. This corrects a previous statement of the author [same J. (2) **3**, 119–129 (1952); these Rev. **14**, 252]. If the condition $(x, y) = 1$ is not required, see Kanagasabapathy, ibid. **3**, 197–205 (1952) [these Rev. **14**, 252]. *L. Tornheim* (Ann Arbor, Mich.).

Davenport, H. Corrigendum to 'Note on a result of Chalk'. Quart. J. Math., Oxford Ser. (2) **5**, 211 (1954).

See same J. **3**, 130–138 (1952); these Rev. **14**, 252.

Cassels, J. W. S. On the product of two inhomogeneous linear forms. J. Reine Angew. Math. **193**, 65–83 (1954).

Let $\xi = \alpha x + \beta y + p$, $\eta = \gamma x + \delta y + q$ be inhomogeneous linear forms in x, y with real coefficients and determinant

$$\Delta = \alpha\delta - \beta\gamma \neq 0.$$

The author investigates the minimum of $|\xi\eta|$ for integer values of x, y under the condition $\xi\eta \neq 0$. Calling the pairs of forms (ξ, η) and (ξ_1, η_1) equivalent if there are integers a, b, c, d, x_0, y_0 and non-zero constants λ, μ such that

$$\xi_1(ax+by+x_0,\ cx+dy+y_0) = \lambda\xi(x, y),$$
$$\eta_1(ax+by+x_0,\ cx+dy+y_0) = \mu\eta(x, y)$$

identically in x, y, then clearly $\mathfrak{M} = |\Delta|^{-1} \min_{\xi\eta\neq 0} |\xi\eta|$ is the same number for two equivalent pairs of forms. By a simple argument the author proves that, for each given constant $\epsilon > 0$, there are indenumerably many inequivalent pairs of forms, not equivalent to homogeneous pairs or to pairs of the type

$$(1) \qquad \xi = x, \quad \eta = y + d \quad (d \text{ arbitrary})$$

such that $\mathfrak{M} \geqq \frac{1}{8} - \epsilon$.

Further, he investigates the pairs with $\mathfrak{M} \geqq \frac{1}{8}$ and succeeds in giving a complete classification of all such pairs, giving for each of them the exact value of $\mathfrak{M}$. His most interesting table is too long to be reproduced here. *J. F. Koksma.*

Samet, P. A. The product of non-homogeneous linear forms. I. Proc. Cambridge Philos. Soc. **50**, 372–379 (1954).

Let θ, ϕ, ψ be the three (real) roots of the cubic equation $t^3-4t+2=0$ and let ξ, η, ζ be the linear forms $x+y\theta+z\theta^2$, $x+y\phi+z\phi^2$, $x+y\psi+z\psi^2$, respectively. For a given point $P(x_0, y_0, z_0)$, $M(P)$ is defined to be the minimum of $|\xi\eta\zeta|$ for all x, y, z for which $x \equiv x_0$, $y \equiv y_0$, $z \equiv z_0$ (mod 1). The author shows that $M(P) \leqq \frac{1}{2}$ which is the best possible refinement of the result obtained by applying Minkowski's theorem for inhomogeneous forms. It is shown that $M(P) = \frac{1}{2}$ only if $P \equiv (0, 0, \frac{1}{2})$ (mod 1). The principal result of the paper is that if $P \not\equiv (0, 0, \frac{1}{2})$ (mod 1), then $M(P) \leqq \frac{1}{4}$. This second minimum is attained only if either $P \equiv (0, \frac{1}{2}, 0)$ or $P \equiv (0, \frac{1}{2}, \frac{1}{2})$ (mod 1). *D. Derry* (Vancouver, B. C.).

Samet, P. A. The product of non-homogeneous linear forms. II. The minimum of a class of non-homogeneous linear forms. Proc. Cambridge Philos. Soc. **50**, 380–390 (1954).

Let θ, ϕ, ψ be the three (real) roots of the equation $t^3+at^2-2t-a=0$, where a is a large positive integer. Let ξ, η, ζ denote the linear forms $x+y\theta+z\theta^2$, $x+y\phi+z\phi^2$, $x+y\psi+z\psi^2$ respectively. For a given point $P(x_0, y_0, z_0)$, $M(P)$ is defined to be the minimum of $|\xi\eta\zeta|$ for all x, y, z for which $x \equiv x_0$, $y \equiv y_0$, $z \equiv z_0$ (mod 1). If $a \equiv 2$ (mod 4) it is shown that $M(P) \leqq a^2/8$ and that $M(P) = a^2/8$ implies $P \equiv (0, 0, \frac{1}{2})$ (mod 1). In the case for which $a \equiv 1$ or 3 mod 4 then $M(P) \leqq (a^2/8)(1-a^{-1})^3$ and $M(P) = (a^2/8)(1-a^{-1})^3$ implies $P \equiv \pm(a^{-1}, 0, \frac{1}{2} - \frac{1}{2}a^{-1})$ (mod 1). *D. Derry.*

Oppenheim, A. Least determinants of integral quadratic forms. Duke Math. J. **20**, 391–393 (1953).

Let g_m be an integral quadratic form in m variables with signature s and determinant $\Delta(g_m)$. The result of this paper

is that

$$\Delta(g_m)\geqq 1/2^m \quad (s\equiv 0), \qquad \Delta(g_m)\geqq 2/2^m \quad (s\equiv \pm 1),$$
$$\Delta(g_m)\geqq 3/2^m \quad (s\equiv \pm 2), \qquad \Delta(g_m)\geqq 4/2^m \quad (s\equiv \pm 3, 4),$$

where all congruences are computed modulo 8. Forms are given for which the minima are attained. The problem is reduced by a series of unimodular transformations to the case where $m=s$, for which case the solution was given by O'Connor and Pall [same J. 11, 319–331 (1944); these Rev. 5, 254]. *D. Derry* (Vancouver, B. C.).

Cohn, Harvey. A periodic algorithm for cubic forms. II. Amer. J. Math. 76, 904–914 (1954).

Continuation of part I [same J. 74, 821–833 (1952); these Rev. 14, 540]; some minor errata to part I are given. In II the computation of units is considered. It is shewn that all units are discoverable by the neighbor processes of I. A hypercomplex notation is introduced. Compared with the Minkowskian algorithm, the present one refines the algebraic properties at the expense of the geometric properties, which also is an advantage with respect to the numerical treatment (the present method is being programmed for the EDVAC at Aberdeen, Md., by Saul Gorn). *J. F. Koksma* (Amsterdam).

Have

***Rademacher, Hans. On Dedekind sums and lattice points in a tetrahedron.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 49–53. Academic Press Inc., New York, 1954. $9.00.

It is proved that if a, b, c are positive integers that are relatively prime in pairs and $s(h, k)$ is the Dedekind sum, then

$$\left(s(bc, a)-\frac{bc}{12a}\right)+\left(s(ca, b)-\frac{ca}{12b}\right)+\left(s(ab, c)-\frac{ab}{12c}\right)$$
$$\equiv -\frac{1}{4}-\frac{1}{12}abc+\frac{1}{12abc} \pmod 2.$$

By a formula of Mordell [J. Indian Math. Soc. (N.S.) 15, 41–46 (1951); these Rev. 13, 322] this is equivalent to

$$N_3(a, b, c)\equiv \tfrac{1}{4}(a+1)(b+1)(c+1) \pmod 2,$$

where $N_3(a, b, c)$ denotes the number of lattice points in the tetrahedron

$$0\leqq x<a, \quad 0\leqq y<b, \quad 0\leqq z<c, \quad 0<\frac{x}{a}+\frac{y}{b}+\frac{z}{c}<1.$$

L. Carlitz (Durham, N. C.).

Wolff, Karl H. Über kritische Gitter im vierdimensionalen Raum (R_4). Monatsh. Math. 58, 38–56 (1954).

In her thesis [Wien, 1944] E. Brunngraber has shown that, if Λ is a critical lattice of a convex symmetrical body K in 4-dimensional space, then it is possible to choose a coordinate system, so that the points $e_1=(1, 0, 0, 0)$, $e_2=(0, 1, 0, 0)$, $e_3=(0, 0, 1, 0)$, $e_4=(0, 0, 0, 1)$ form a basis of Λ while the first three of these points, together with at least one of the points $p_1=(0, 0, 0, 1)$, $p_2=(0, 1, 1, 2)$, $p_3=(1, 1, 1, 2)$, $p_4=(1, 1, 1, 3)$, $p_5=(1, 1, 2, 4)$, $p_6=(1, 2, 2, 5)$, lie on the boundary of K. The author gives another proof of this result. He is then able to list, in each of the cases $k=1, 2, \cdots, 6$, a finite number of points with integral coordinates, having the property, that, if none of these points lie in a convex symmetrical body with e_1, e_2, e_3, p_k on its boundary, then no point with integral coordinates, other than the origin, lies in the body. This extends to 4-dimensions classical results of Minkowski [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1904, 311–355] proved in explicit form only in 2 and 3 dimensions. The method is applied, in a refined form to the example when K is a sphere; and the critical lattices are completely determined in this case. *C. A. Rogers* (Birmingham).

Szüsz, Petér. Sharpening of a theorem of Hardy and Littlewood. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei 4, 205–208 (1954). (Hungarian)

Hungarian version of Acta Math. Acad. Sci. Hungar. 4, 115–118 (1953); these Rev. 15, 293.

ANALYSIS

Bajraktarević, Mahmud. Sur quelques cas spéciaux du théorème généralisé de la moyenne. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 115–128 (1953). (Serbo-Croatian. French summary)

For a sufficiently smooth function $\xi(x_1, x_2)$, the author has previously [Bull. Soc. Math. Phys. Serbie 3, nos. 3–4, 15–23 (1951); these Rev. 14, 625, 1278] determined three conditions which are both necessary and sufficient in order that there should exist functions $f(x)$ and $\varphi(x)$ for which

$$(*) \qquad \frac{f(x_2)-f(x_1)}{\varphi(x_2)-\varphi(x_1)}=\frac{f'(\xi)}{\varphi'(\xi)} \quad (x_1\leqq\xi\leqq x_2).$$

Two of these conditions are $\xi(x_1, x_2)=\xi(x_2, x_1)$ and $\xi(x, x)=x$; the third condition involves first- and second-order partial derivatives of ξ. By a change of variables, the author now transforms the third condition to a form which is sometimes more readily applicable to mean-value problems, for example to the determination of a function $\psi(x)$ for which both (*) and $\psi(\xi)=\frac{1}{2}(\psi(x_1)+\psi(x_2))$ are satisfied.

E. F. Beckenbach (Los Angeles, Calif.).

Wright, E. M. An inequality for convex functions. Amer. Math. Monthly 61, 620–622 (1954).

Let $f(x)$ denote a real-valued function defined on the half-line $x\geqq 0$. The following proposition is denoted by $P(m, f)$: For every set of m real numbers $a_1, \cdots, a_m$ with $a_1\geqq a_2\geqq \cdots \geqq a_m\geqq 0$,

$$f(a_1)-f(a_2)+\cdots+f(a_m)\geqq f(a_1-a_2+\cdots+a_m) \quad (m \text{ odd}),$$
$$f(a_1)-f(a_2)+\cdots-f(a_m)\geqq f(a_1-a_2+\cdots-a_m)-f(0) \quad (m \text{ even}).$$

Weinberger and Bellman [Weinberger, Proc. Nat. Acad. Sci. U. S. A. 38, 611–613 (1952); these Rev. 14, 24] have proved $P(m, f)$ for $f(x)=x^r$ $(r\geqq 1)$ and for continuously differentiable convex f respectively. The author proves by induction that, for every f, $P(m, f)$ is equivalent to $P(3, f)$. Also $P(3, f)$ implies convexity in the sense that

$$f(\tfrac{1}{2}(x_1+x_2))\leqq \tfrac{1}{2}(f(x_1)+f(x_2))$$

for all non-negative x_1, x_2. For continuous f, $P(3, f)$ is equivalent to convexity. *F. F. Bonsall.*

Fan, Ky, and Lorentz, G. G. An integral inequality. Amer. Math. Monthly 61, 626–631 (1954).

If f and g are real functions on $[0, 1]$, then $f \dashv g$ is defined to mean $\int_0^x f(t)dt \leqq \int_0^x g(t)dt$ for $0 \leqq x \leqq 1$, with equality at $x=1$. The problem is to find conditions on Φ which assure that

$$\int_0^1 \Phi(t, f_1, \cdots, f_n)dt \leqq \int_0^1 \Phi(t, g_1, \cdots, g_n)dt$$

for every set of bounded functions f_i, g_i, such that $f_i \dashv g_i$. Necessary and sufficient conditions in terms of mixed second differences of Φ are found. Special attention is paid to the case in which f_i and g_i are decreasing functions.

W. Rudin (Rochester, N. Y.).

Morgenstern, Dietrich. Unendlich oft differenzierbare nicht-analytische Funktionen. Math. Nachr. 12, 74 (1954).

The author establishes very concisely, by a category argument, the existence of nowhere analytic infinitely differentiable functions. *R. P. Boas, Jr.*

Lişcu, Traian. Sur le critère d'unicité dans le problème des moments. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 2, 495–557 (1951). (Romanian. Russian and French summaries)

The author sets out to discover whether Hamburger's uniqueness criterion for his moment problem (in the form of the vanishing of the limit of a quotient of determinants) implies the determinant criterion for the existence of a solution of the moment problem. By explicit calculation with four sequences associated with the classical orthogonal polynomials, he shows that it does not. The paper contains many calculations of various quantities associated with the moments and with the orthogonal polynomials.

R. P. Boas, Jr. (Evanston, Ill.).

Calculus

Have

✶Ostrowski, A. Vorlesungen über Differential- und Integralrechnung. Dritter Band. Integralrechnung auf dem Gebiete mehrerer Variablen. Verlag Birkhäuser, Basel-Stuttgart, 1954. 475 pp. Broschiert, SFr. 73.85; ganzleinen, SFr. 78.00.

This third and final volume completes the author's treatise on the differential and integral calculus which doubtless is one of the most comprehensive on the subject. The objective of the author has been discussed in the reviews of the earlier volumes [these Rev. 13, 540]. The present volume is concerned principally with the integral calculus and Fourier analysis. The exposition is meticulous. The careful account of multiple integral theory takes some 180 pages. The problem collection is particularly rich. A summary of the contents follows. Ch. 1, Supplementary material on indefinite integrals. Ch. 2, The concept of the multiple integral. Ch. 3, The calculation of multiple integrals. This chapter includes line integrals, total differentials, change of variable in double integrals. Ch. 4, Applications of multiple integrals; surface area, Ostrogradski's theorem, Stokes' theorem. Ch. 5, Improper simple integrals. Ch. 6, Improper multiple integrals and the gamma function. Ch. 7, Fourier series and integral. *M. Heins.*

✶Schmidt, Harry. Einführung in die Vektor- und Tensorrechnung unter besonderer Berücksichtigung ihrer physikalischen Bedeutung. VEB Verlag Technik, Berlin, 1953. 116 pp.

Starting with a discussion of real and complex numbers, the first two chapters give a clear exposition of the elements of vector algebra and calculus with some physical applications, the terminal point being the solution of rot $\mathbf{u}=\mathbf{w}$ and the integral solution of Poisson's equation. With the stress tensor as an illustration, the third chapter deals briefly with tensors as vector triples and as matrices. Indicial notation is not used. *J. L. Synge* (Dublin).

✶Kil'čevskiĭ, N. A. Èlementy tenzornogo isčisleniya i ego priloženiya k mehanike. [Elements of tensor calculus and its applications to mechanics.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 167 pp. 5.15 rubles.

This diminutive booklet somehow contrives to cover more material than other books twice its size. For instance, it includes the representation of the motion of a system of rigid bodies as the motion of a particle in a Riemannian space in nonholonomic constraints, and the invariant form of the equations of motion of an elastic continuum. Whether this feat has been accomplished at the expense of clarity can be judged only by a reader not familiar with the subject. A sampling of the book did not reveal any obscure or loose statements. The brevity is partly achieved by omitting problems, exemplifications, and informal remarks aimed at giving mental comfort to the reader. *A. W. Wundheiler.*

✶Narayan, Shanti. A text book of vector algebra (with applications). S. Chand & Co., Delhi, 1954. iv+190 pp. Rupees 5/4/0.

This text treats vector algebra with extensive applications to geometry and statics. Implicitly assuming a three-dimensional real space and making extensive use of the many figures, the non-metric vector relations are developed in chapters one and two before metric properties are introduced in chapter three. After the rules for vector products are fully developed, the whole machinery is put to work to obtain the main results of linear analytic geometry. The final chapter notes rather carefully the need for "line," as opposed to the previously used "free," vectors in describing forces and obtains the basic theorems of statics.

As the author notes in the introduction, the motivation for the study of vectors in abstract algebra is quite different but within the assigned purpose the exposition, descriptive rather than deductive, is quite clear. A large number of fully worked examples and nearly two hundred exercises, with answers where required, add to the value of the book. Not representative of the care with which the book has been written is the error in example 4 on page 126 where the unit vectors chosen are not orthonormal as implied.

A second volume dealing with the differential and integral calculus of vectors and their applications is promised.

W. Givens (Princeton, N. J.).

✶Mitrinović, Dragoslav S. Zbirka zadataka iz matematike za studente tehničkih i prirodno-matematičkih fakulteta. [Collection of problems in mathematics for students of engineering and science-mathematics faculties.] Znanje, Belgrade, 1954. 175 pp.

A collection of problems assembled from a variety of sources and concerning elementary topics in analytic geometry, differential and integral calculus, series, differential equations, functions of a complex variable, etc.

★**Munthe Hjortnaes, Margrethe. Transformation of the series $\sum_{k=1}^{\infty} 1/k^3$ to a definite integral.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 211–213 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution). (Norwegian)

The result is:

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = 10 \int_0^c t^2 \coth t \, dt, \quad c = \log \tfrac{1}{2}(1+\sqrt{5}).$$

Theory of Sets, Theory of Functions of Real Variables

Schmidt, Jürgen. Abgeschlossenheits- und Homomorphiebegriffe in der Ordnungstheorie. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **3**, 223–225 (1954).

Considérations générales sur la notion d'homomorphismes de structures. Définition et étude de classes d'homomorphismes d'un ensemble ordonné dans un autre, plus restreintes que la classe des applications croissantes.

P. Samuel (Clermont-Ferrand).

Neumer, Walter. Zur Konstruktion von Ordnungszahlen. IV. Math. Z. **61**, 47–69 (1954).

The author shows, as promised in part I [Math. Z. **58**, 391–413 (1953); these Rev. **15**, 512], the connection between his constructive algorithm developed there and Ackermann's [ibid. **53**, 403–413 (1951); these Rev. **12**, 579] system of notation for ordinal numbers. *F. Bagemihl.*

Amato, Vincenzo. Ipotesi del continuo. Matematiche, Catania **8**, no. 2, 49–50 (1953).

Brief expository remarks. *F. Bagemihl.*

Sierpinski, W. Sur un théorème de recouvrement équivalent à un cas particulier de l'axiome du choix. Ganita **4**, 155–158 (1953).

It is shown (without the use of the axiom of choice) that the following two propositions are equivalent: (1) [Veress, Acta Litt. Sci. Szeged. **6**, 34–45 (1932).] If

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots, \quad T \subset \bigcup_{n=1}^{\infty} E_n,$$

and for every infinite sequence of distinct elements of T there exists a natural number k such that E_k contains infinitely many terms of this sequence, then there exists a natural number m such that $T \subset E_m$. (2) If $H_1, H_2, \cdots, H_n, \cdots$ is an infinite sequence of nonempty sets such that $H_1 \supset H_2 \supset \cdots \supset H_n \supset \cdots$, then there exists an infinite sequence $p_1, p_2, \cdots, p_n, \cdots$ such that $p_n \in H_n$ $(n=1, 2, 3, \cdots)$.

F. Bagemihl (Princeton, N. J.).

Sierpiński, W. Sur une proposition équivalente à l'existence d'un ensemble de nombres réels de puissance $\aleph_1$. Bull. Acad. Polon. Sci. Cl. III. **2**, 53–54 (1954).

It is shown, without the use of the axiom of choice, that the following two propositions are equivalent. (1) There exists a function f which assigns to every enumerable set E of real numbers a real number $f(E)$ non-ε E. (2) There exists a set, of power $\aleph_1$, of real numbers. *F. Bagemihl.*

Popruženko, J. Sur le phénomène de convergence de M. Sierpiński. Fund. Math. **41**, 29–37 (1954).

Let $\{f_n(x)\}$ be a sequence of real-valued functions which converges for every element x of a set E; $\{f_n(x)\}$ is said to converge σ-uniformly on a subset M of E if $M = \bigcup_{k=1}^{\infty} M_k$ and $\{f_n(x)\}$ converges uniformly on M_k $(k=1, 2, 3, \cdots)$. Assuming the continuum hypothesis, Sierpiński [cf. Hypothèse du continu, Warszawa-Lwów, 1934, p. 52] showed that there exists a convergent sequence of functions of a real variable, which converges nonuniformly on every nonenumerable set of real numbers. The present author proves, without the aid of the continuum hypothesis, that there exists one and only one cardinal number $\mathfrak{m}$ such that on every set E of power $\mathfrak{m}$ there exists at least one convergent sequence $\{f_n(x)\}$ which is not σ-uniformly convergent on any subset of E of the same power, whereas every convergent sequence on E is σ-uniformly convergent on every subset of E of smaller power; moreover, $\mathfrak{m}$ is equal to the regular aleph $\aleph_\tau$ defined by Rothberger [Fund. Math. **32**, 294–300 (1939)] $(\aleph_1 \leq \aleph_\tau \leq 2^{\aleph_0})$. This result is then applied to abstract measure theory to establish the existence of an enumerable family of absolutely incommensurable subsets of any set of power $\aleph_\tau$. Finally, the author remarks that the existence of a convergent sequence $\{f_n(x)\}$ on a nonenumerable set that converges nonuniformly on every nonenumerable subset is equivalent to the existence of (Ω, ω^*)-gaps in the family of all sequences of natural numbers.

F. Bagemihl (Princeton, N. J.).

Popruženko, J. Sur une décomposition des ensembles indénombrables. I. Fund. Math. **41**, 146–149 (1954).

Consider the following proposition (P): If E is of power $\mathfrak{m}$, there exists a double sequence $\{A_m{}^i\}$ of subsets of E such that $E = \bigcup_{m=1}^{\infty} A_m{}^i$ $(i=1, 2, 3, \cdots)$; for every i, the sets $A_m{}^i$ $(m=1, 2, 3, \cdots)$ are mutually exclusive; and, for every sequence of natural numbers $\{m_i\}$, $\bigcap_{i=1}^{\infty} \bigcup_{j=1}^{m_i} A_j{}^i$ is at most enumerable. Assuming the continuum hypothesis, Banach and Kuratowski [Fund. Math. **14**, 127–131 (1929)] showed that (P) holds for $\mathfrak{m} = 2^{\aleph_0}$. The present author proves (without the aid of the continuum hypothesis) that in order that there exist at least one cardinal number $\mathfrak{m}$ greater than $\aleph_0$ and satisfying (P), it is necessary and sufficient that there exist a set of real numbers, of power $\aleph_1$, possessing property λ and lacking property λ' (S has property λ, if every one of its enumerable subsets is a relative G_δ; T has property λ', if, for every enumerable set X of real numbers, $T \cup X$ has property λ). *F. Bagemihl* (Princeton, N. J.).

Császár, Ákos. Sur la structure des ensembles de niveau des fonctions réelles à deux variables. Acta Sci. Math. Szeged **15**, 183–202 (1954).

Let $\mathfrak{N}$ denote any family of plane sets having the following sole properties: (1) $B \subset A$, $A \in \mathfrak{N}$ implies $B \in \mathfrak{N}$; (2) $A_i \in \mathfrak{N}$, $i=1, 2, \cdots$, implies $\sum A_i \in \mathfrak{N}$. Thus $\mathfrak{N}$ may be the family of all plane sets of measure zero, or of linear measure zero, or empty, etc. Denote by $\mathfrak{H}$ the family of all plane sets which can be covered by a countable family of rectifiable arcs, and by $\mathfrak{N}^*$ the family of all sets $N+H$ with $N \in \mathfrak{N}$, $H \in \mathfrak{H}$. Let $f(z)$ be a real function defined for all complex z. Denote by $S(z_0, r, \alpha, \beta)$ the set of all $z = z_0 + \rho \exp(i\varphi)$ with $0 < \rho < r$, $\alpha < \varphi < \beta$. Then $f(z)$ is said to have the property $C^*_{\alpha\beta}$ at $z = z_0$ if $E[f(z) < f(z_0), z \in S(z_0, r, \alpha, \beta)] \in \mathfrak{N}$ for some $r > 0$, while the same set where $<$ is replaced by $\leq$, does not belong to $\mathfrak{N}$ for any r. The property $D^*_{\alpha\beta}$ is defined analogously where $<$, $\leq$ are replaced by $>$, $\geq$. Properties C, D and other ones considered by the author concern the distribution of the values taken by $f(z)$. The author shows that certain combinations are somehow exceptional. Here is one of the statements: If $\alpha < \beta$, $\gamma < \delta$ are given, and $f(z)$ has both properties $C^*_{\alpha\beta}$, $D^*_{\gamma\delta}$ at the points of a set E, then

$E=N+L$, where $N \in \mathfrak{N}^*$, and there is a countable family of circles K and of sets $L \subset K$, $L \in \mathfrak{N}$, such that $L \subset \sum K$, and and f is constant on each $K-L$. Other properties are given relating the question of the distribution of the values of $f(z)$ with the Lebesgue density theorem. *L. Cesari.*

Sion, Maurice. On the existence of functions having given partial derivatives on a curve. Trans. Amer. Math. Soc. **77**, 179–201 (1954).

It has been proved by A. P. Morse [Ann. of Math. (2) **40**, 62–70 (1939)] that, if A is a connected subset of E_n (Euclidean space of n dimensions, $n>1$) and if f is a function on E_n such that all its first partial derivatives vanish at all points of A, then f is constant on A provided that $f \in C^n$ (that is, when f has continuous nth partial derivatives). H. Whitney [Duke Math. J. **1**, 514–517 (1935)] has shown that f need not be constant if it is assumed only that $f \in C^{n-1}$. This paper deals with a similar problem in which all the second partial derivatives of f vanish on A, and all the first partial derivatives vanish at some point of A. The result of A. P. Morse shows that f is constant on A if $f \in C^{n+1}$. The author finds sufficient conditions for the existence of functions having given partial derivatives on A, and extends the construction of H. Whitney to show that, if it is supposed only that $f \in C^n$, then f need not be constant on A. *U. S. Haslam-Jones* (Oxford).

Sengupta, H. M. On continuous semi-independent functions. Quart. J. Math., Oxford Ser. (2) **5**, 172–174 (1954).

The author's definition of semi-independence is not quite clear to the reviewer. A possible interpretation is this: two real-valued, continuous functions f and g on the closed unit interval are semi-independent if, for each value that f assumes, the restriction of g to the set where f assumes that value has the same range as g, and if the same holds with f and g interchanged. The author's result is that if f and g are real-valued, non-constant, continuous functions on the closed unit interval, then a necessary and sufficient condition that the range of the mapping $t \rightarrow (f(t), g(t))$ $(0 \leq t \leq 1)$ be a rectangle is that f and g be semi-independent. The result is a trivial consequence of the definition given above. *P. R. Halmos* (Chicago, Ill.).

Volkmann, Bodo. Zwei Bemerkungen über pseudorationale Mengen. J. Reine Angew. Math. **193**, 126–128 (1954).

A is a sequence $\{\rho_n\}$, $0<\rho_n \leq 1$. $\rho_n = \sum e_{in} g^{-i}$ is the g-adic development of ρ_n. A^* is the set of numbers $\rho = \sum e_i g^{-i}$ such that, for each $\epsilon>0$, $\exists p, q, e_{ip} \leq e_i \leq e_{iq}$,

$$\liminf \left[n^{-1} \sum_{i=1}^{n} (e_{iq} - e_{ip})\right] < \epsilon.$$

The author proves that A^* is of Hausdorff dimension 0. When $g=2$ and A consists of the rational fractions, A^* is the set of pseudorational fractions [cf. Volkmann, same J. **190**, 199–230 (1952); these Rev. **15**, 15]. His second remark is that the Schnirelmann sum of two pseudorational sequences of integers is not necessarily a pseudorational sequence. *H. D. Ursell* (Leeds).

Besicovitch, A. S., and Taylor, S. J. On the complementary intervals of a linear closed set of zero Lebesgue measure. J. London Math. Soc. **29**, 449–459 (1954).

Let $a=\{a_n\}$ be any decreasing sequence of positive numbers such that $\sum a_n = 1$, and let E be a closed subset of [0, 1] whose complementary open intervals have lengths equal to the a_n when arranged in decreasing order. The authors investigate relationships between the Hausdorff dimension and measure of E and properties of a. Setting $\lambda(\beta, a) = \liminf_{n\to\infty} n(r_n/n)^\beta$, where $r_n = \sum_n^\infty a_j$ and $0<\beta \leq 1$, and setting $\alpha(a) = \liminf_{n\to\infty} \alpha_n$, where α_n is defined by $n(r_n/n)^{\alpha_n} = 1$, the authors prove that $\Lambda^\beta E \leq \lambda(\beta, a)$ and that $\dim E \leq \alpha(a)$. Conversely, it is shown that if $0 \leq \beta \leq \alpha(a)$, then there exist sets E associated with a of dimension β, and furthermore, if $\beta>0$ and if $0 \leq \gamma \leq \lambda(\beta, a)/4$, then there exists such a set E with $\Lambda^\beta(E) = \gamma$. *L. H. Loomis.*

Eggleston, H. G. Two measure properties of Cartesian product sets. Quart. J. Math., Oxford Ser. (2) **5**, 108–115 (1954).

The author proves that if E is a measurable plane set of Lebesgue measure 1 included in the unit square $S=I \times J$, then there exists a non-void perfect set P included in the unit interval I, and a set Q of positive linear measure included in the unit interval J, such that $P \times Q \subset E$. It is also shown that if the complement $S-E$ has finite linear measure, then P and Q can both be chosen to have positive linear measure, whereas there exists a set E whose complement has countable infinite linear measure for which no such pair of positive sets exists. *L. H. Loomis.*

Enomoto, Shizu. Notes sur l'intégration. I. Quelques propriétés des fonctions d'intervalle. Proc. Japan Acad. **30**, 176–179 (1954).

In this first of three notes the author studies an extension of Denjoy's complete totalisation to the Euclidean space E_n $(n>1)$, thus associating, in addition to the original fundamental researches of Denjoy, with some of the researches of Krzyżański, Ridder, Kempisty, and Romanowski. This is based largely on the use of additive functions of interval (the author considers an interval to be closed). The three notes constitute essentially an abstract of a theory developed by the author. A detailed description of the definitions and results would not be feasible. On an interval R (of E_n) a certain class D_R is defined, consisting of functions $f(p)$ of point p. The definition D_R is such that with respect to D_R the following can be shown. If $F(I)$ is an additive interval-function $(I \leq R)$ and the strong derivative $F_s'(p)$ exists in R, then $f(p) = F_s'(p) \in D_R$; if R is an interval in E_1, D_R is identical with functions completely totalisable in the sense of Denjoy; $F(I)$ involved in the definition of D_R is uniquely determined by $f(p)$; $F'(p)$ exists almost everywhere in R and equals $f(p)$; if f_1, f_2 are in D_R, f_1+f_2 will be in D_R and for the corresponding interval-functions one will have $F(I) = F_1(I) + F_2(I)$ $(I \leq R)$; if $f(p)$ is in D_R and $g(p) = f(p)$ a.e., then g is in D_R and, if $F(I)$ is associated with $f(p)$ (according to the definition of D_R), then $F(I)$ will be associated with $g(p)$. *W. J. Trjitzinsky* (Urbana, Ill.).

Enomoto, Shizu. Notes sur l'intégration. II. Une propriété du recouvrement fermé de l'intervalle. Proc. Japan Acad. **30**, 289–290 (1954).

In this second note of the series of three the author states a property, essential for the sequel, of a sequence of closed sets covering an interval in E_n. The formulation of the covering theorem is quite involved and could not be given here within a space less than that of the note itself. *W. J. Trjitzinsky* (Urbana, Ill.).

Enomoto, Shizu. Notes sur l'intégration. III. Théorème de Fubini. Proc. Japan Acad. 30, 437–442 (1954).

In this final note the author actually gives an extension to E_2 of the Denjoy complete total. The presentation of the theory is descriptive rather than constructive. The total characterized in E_2 has the following properties. (1) If $f(x, y)$ is totalisable on an interval R and $F(I)$ is the total of $f(x, y)$ for $I<R$, then $F'(x, y)$ (a suitable derivative) $=f(x, y)$ almost everywhere on R. (2) The Fubini theorem holds for totals. (3) If $F(I)$ is an additive function of interval $I\leqq R$, having $F_s'(x, y)$ at every (x, y) in R, then $F_s'(x, y)$ is totalisable and $F(R)$ is the total of $F_s'(x, y)$. *W. J. Trjitzinsky* (Urbana, Ill.).

Dživarševišvili, A. G. On Fubini's theorem for the double Denjoy integral. Soobščeniya Akad. Nauk Gruzin. SSR 14, 393–398 (1953). (Russian)

Let $f(x, y)$ be a function defined in the rectangle

$$R_0=[(a, b)(c, d)],$$

such that the double Denjoy integral

$$T(x, y)=\int_a^x\int_c^y f(t, \tau)dtd\tau$$

exists in the sense of Čelidze [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 15, 155–242 (1947); these Rev. 14, 735]. Suppose that the iterated integrals

$$F(x, y)=\int_a^x dt\int_c^y f(t, \tau)d\tau, \quad \psi(x, y)=\int_c^y d\tau\int_a^x f(t, \tau)dt$$

exist (in the Denjoy-Khinchin sense). The author gives several conditions for the truth of the formula

$$(*)\qquad T(\beta, \delta)=F(\beta, \delta)=\psi(\beta, \delta)$$

for any $\beta\leqq b$, $\delta\leqq d$. For instance, Theorem 3: Let $F(x, y)$ and $\psi(x, y)$ be (ACG) on R_0 in the sense of Čelidze; if $R_0=\sum_1^\infty E_k$ and the derivatives $D_{E_k}F(x, y)$, $D_{E_k}\psi(x, y)$ exist a.e. for each $k=1, 2, \cdots$, then (*) is true. The author also gives some conditions under which $P(x, y)dx+Q(x, y)dy$ is a total differential, assuming that $\partial P/\partial y=\partial Q/\partial x$ almost everywhere. For instance, Theorem 8: If the function $f(x, y)=\partial P/\partial y=\partial Q/\partial x$ satisfies the hypothesis of Theorem 3, then there exists a function $H(x, y)$ such that $\partial H/\partial x=P$, $\partial H/\partial y=Q$. These theorems are based on previous results due to Tolstov [Trudy Mat. Inst. Steklov. 35 (1950); these Rev. 13, 448]. No proofs are given. *M. Cotlar.*

Hadwiger, H. Über additive Funktionale k-dimensionaler Eipolyeder. Publ. Math. Debrecen 3 (1953), 87–94 (1954).

A functional $\varphi(P)$ defined in the set of closed convex polyhedra in k-dimensional euclidean space is called additive if $\varphi(P)+\varphi(Q)=\varphi(P+Q)+\varphi(PQ)$, when $P+Q$ is a convex polyhedron divided into P and Q by a $(k-1)$-dimensional polyhedron PQ. It is called simply additive if $\varphi(P)+\varphi(Q)=\varphi(P+Q)$, i.e., if it is additive and vanishes for polyhedra of dimension less than k. The main purpose of the paper is to show that any additive functional may be expressed in the form $\varphi(P)=\sum_{i=0}^k \chi_i(P)$, where $\chi_i(P)$ is derived from a simply additive functional in i-dimensional space through a certain formula (involving summation over the i-dimensional edges of P). The paper ends with some remarks on functionals invariant under movements. The problem is formulated whether the volume is the only simply additive and invariant functional which is bounded in the set of polyhedra contained in an arbitrary cube. *B. Jessen* (Copenhagen).

Hadwiger, H. Deckungsäquivalenz und Zerlegungsäquivalenz bei Funktionen in abstrakten Räumen und invariante Integration. Arch. Math. 5, 115–122 (1954).

This paper continues investigations by Kirsch [Math. Ann. 124, 343–363 (1952); these Rev. 14, 28] and by Hadwiger and Kirsch [Portugaliae Math. 11, 57–67 (1952); these Rev. 14, 147]. In a space R in which an abelian group Γ of one-to-one transformations and a unit set with characteristic function $E(x)$ are given, two functions $F(x)$ and $G(x)$ are called "deckungsäquivalent", if to each $\epsilon>0$ there exist elements $\alpha_\nu, \beta_\nu, \gamma_\mu$ of Γ such that

$$\left|\sum_1^n F(\alpha_\nu x)-\sum_1^n G(\beta_\nu x)\right|\leqq\sum_1^m E(\gamma_\mu x),$$

where $m/n<\epsilon$, and "zerlegungsäquivalent", if to each $\epsilon>0$ there exist decompositions $F=\sum_1^m F_\nu$, $G=\sum_1^m G_\nu$ and elements α_ν, β_ν of Γ such that $|\sum_1^m F_\nu(\alpha_\nu x)-\sum_1^m G_\nu(\beta_\nu x)|\leqq\epsilon E(x)$ for all $x\,\epsilon\, R$. These two notions are shown to be identical. Applications to abstract integration are announced. *B. Jessen* (Copenhagen).

Lorch, Edgar Raymond. Su certe estensioni del concetto di volume. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 16, 25–29 (1954).

Dans un espace à n dimensions: $x=(x_1, \cdots, x_n)$, soit K un corps convexe dont la frontière est définie par $\varphi(x)=1$, où φ est une fonction positive et positivement homogène de degré 1, admettant des dérivées partielles continues jusqu'au second ordre, soit K^* le conjugué de K; on pose:

$$V_r(K^*)=\left[\frac{1}{n}\int_\Omega \varphi^{-nr}(x)C(x)d\Omega\right]^{1/r}\qquad(1\leqq r\leqq\infty)$$

où $C(x)$ est une fonction bien déterminée, formée à partie de φ et de ses dérivées partielles jusqu'au second ordre, où Ω est la sphère de rayon unité, $d\Omega$ son élément d'aire; en faisant $r=1$, on a le volume ordinaire de K^*. L'auteur propose, en conséquence, d'appeler l'expression précédente volume d'ordre r de K^*; l'expression V_∞ est très simple; le calcul de V_2 est fait pour l'ellipsoïde. *J. Favard.*

Fichera, Gaetano. Sulla derivazione delle funzioni additive d'insieme. Rend. Sem. Mat. Univ. Padova 23, 366–397 (1954).

A theory of derivation of additive set functions of bounded variation is presented here. In §1 the author gives definitions and properties of systems of sets and of measure and integration, corresponding to his recent work, "Lezioni sulle trasformazioni lineari" [vol. I, Ist. Mat., Univ., Trieste, 1954]. In particular, he calls a family $\mathfrak{R}$ of abstract sets a "semi-ring" (semi-anello) if 1) $\mathfrak{R}$ is closed with respect to intersection and 2) for any two sets I', I'' of $\mathfrak{R}$ with $I'\subset I''$ there exist finitely many disjoint sets I_k $(k=1, 2, \cdots, n)$ of $\mathfrak{R}$ such that $I''=I'+I_1+\cdots+I_n$ and, for each $h\leqq n$, the set $I'+I_1+\cdots+I_h$ belongs also to $\mathfrak{R}$. A completely additive set function μ of bounded variation defined in $\mathfrak{R}$ is called a measure. Let now μ be a non-negative measure in the semi-ring $\mathfrak{R}$, let $F(I)$ be any non-negative, additive set function defined in $\mathfrak{R}$, and let $I_0\,\epsilon\,\mathfrak{R}$. Then in §2 the author defines the derivative of F with respect to μ on I_0 in the following manner: He designates by $\Theta[F, I_0]$ the class of all real functions f defined in I_0 such that (a) f is non-negative and μ-summable in I_0, (b) for every $I\,\epsilon\,\mathfrak{R}$, contained in I_0, one has $\int_I fd\mu\leqq F(I)$. He now considers the functional $J_{I_0}(f)=\int_{I_0} fd\mu$ for every f of the class $\Theta[F, I_0]$. If $J_{I_0}(f)$ has a maximum in this class, then he calls F

"μ-derivable" on I_0 and he calls the function f which gives that maximum of $J_{I_0}(f)$ in $\Theta[F, I_0]$ the "μ-derivative" F' of F on I_0. This μ-derivative is determined except for the addition of a function which is μ-equivalent to zero in I_0. In §3 the μ-derivative is generalized to functions F (not necessarily non-negative) which are assumed to be additive and of bounded variation. This is done by representing F as the difference of its positive and negative variations. Then in §4 the Radon-Nikodým theorem is proved for additive functions F of bounded variation, defined in $\mathfrak{R}$. Finally, in §5 the Lebesgue decomposition for such functions F (on any $I_0 \in \mathfrak{R}$) is obtained. *A. Rosenthal* (Lafayette, Ind.).

Gehring, F. W. A study of α-variation. I. Trans. Amer. Math. Soc. **76**, 420–443 (1954).

Let f be a real- or complex-valued function on $a<x<b$ and let $\alpha=1/p$ where $p\geqq 1$. In the later 1930's, in a series of papers, some joint with E. R. Love, the reviewer studied and employed the notion of pth power variation of f, a notion previously introduced when $p=2$ by N. Wiener. The author denotes by W_α the class of functions for which this variation is finite and obtains in terms of moment constants, necessary and sufficient conditions for a function to belong to this class. He then proceeds to the major contribution of the paper which consists in solving a Faltung problem raised by J. E. Littlewood and the reviewer. Applications are made to the theory of infinite series: these concern a notion of summability (by Abel or various Cesàro methods) intermediate between ordinary summability and absolute summability. Tauberian and other theorems are established for this notion. *L. C. Young* (Madison, Wis.).

Kašanin, R. Les intégrales des fonctions différentiables. Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 29–44 (1953). (Serbo-Croatian. French summary)

The author gives a proof of Green's theorem for functions $f(x, y)$, $g(x, y)$ that are differentiable in the sense of Stolz and of other elementary theorems involving partial derivatives. All these theorems are known to hold under much more general conditions. *M. Golomb.*

Theory of Functions of Complex Variables

✶Caratheodory, C. Theory of functions of a complex variable. Vol. 2. Translated by F. Steinhardt. Chelsea Publishing Company, New York, 1954. 220 pp.

Translation of the author's Funktionentheorie, Band 2 [Birkhäuser, Basel, 1950; these Rev. **12**, 248].

Stone, Marshall H. The introduction to the theory of analytic functions. Bol. Soc. Mat. Mexicana **10**, nos. 3–4, 29–30 (1953).

The author suggests that a study of the primitives of the equation $dw/dz=f(z)$, where $f(z)$ is continuous and has a derivative in a domain D, simplifies the introduction of the Cauchy theory. *A. J. Lohwater* (Ann Arbor, Mich.).

Müller, Claus. Über die Umkehrung des Cauchyschen Integralsatzes. Arch. Math. **6**, 47–51 (1954).

Let $\{C_n\}$ be a sequence of regular curves whose interiors G_n have area F_n. The sequence $\{C_n\}$ is called a null-sequence if, for every $\epsilon>0$, there is an integer $N(\epsilon)$ such that all regions G_n with $n\geqq N(\epsilon)$ lie inside the circle $|z|<\epsilon$. Let $f(z)$ be continuous in the plane, and let $\{C_n\}$ be a fixed null-sequence. If, for every convergent sequence $\{z_n\}$ lying in a given finite region G, $\oint_{C_n} f(z_n+\zeta)d\zeta=o(F_n)$, where the integrals are Cauchy integrals of the complex variable ζ, then $f(z)$ is analytic in G. *A. J. Lohwater.*

Pogorzelski, W. Problème non linéaire d'Hilbert pour le système de fonctions. Bull. Acad. Polon. Sci. Cl. III. **2**, 3–5 (1954).

In the complex plane let $L_0, L_1, \cdots, L_p$ be closed contours (suitably regular simple curves) without common points, L_0 containing $L_1, \cdots, L_p$, the L_j $(1\leqq j\leqq p)$ enclosing disjoint domains S_j^-, respectively; S_0^- is the infinite domain bounded by L_0 and S^+ is the domain bounded by $L_0, \cdots, L_p$; $L=L_0+\cdots+L_p$. Summary indications are given as to the ways of finding functions $\Phi_1(z), \cdots, \Phi_m(z)$, analytic in $S^+, S_0^-, \cdots, S_p^-$ and satisfying:

$$\Phi_j^+=G_j\Phi_j^-+\lambda F_j[t, \Phi_1^+, \cdots, \Phi_m^+, \Phi_1^-, \cdots, \Phi_m^-]$$

(t on L; $j=1, \cdots, m$; λ a parameter), where the $G_j(\neq 0)$ are analytic in a neighborhood of $L_0+\cdots+L_p$, the same being true of the $F_j(z_1, u_1, \cdots, u_{2m})$ considered as functions of z, $|u_\nu|\leqq R$ $(\nu=1, \cdots, 2m)$. *W. J. Trjitzinsky.*

Putnam, C. R. Remarks on periodic sequences and the Riemann zeta-function. Amer. J. Math. **76**, 828–830 (1954).

A theorem is proved of which the gist is that the imaginary part of $\zeta(\sigma+it)$, on any fixed line $\sigma=$const. >0, does not tend to zero as $t=t_n\to\infty$, if the sequence $t_1, t_2, \cdots$ approximates closely enough to a periodic sequence $d, 2d, 3d, \cdots$. In particular, the sequence of the consecutive zeros, of $\zeta(\frac{1}{2}+it)$ does not contain any periodic subsequence.
E. C. Titchmarsh (Oxford).

Schottlaender, Stefan. Der Hadamardsche Multiplikationssatz und weitere Kompositionssätze der Funktionentheorie. Math. Nachr. **11**, 239–294 (1954).

This dissertation deals with the compositions of singularities associated with various compositions of functions. After a historical and critical survey, the author proves in great detail Hadamard's composition theorem for the special case where the functions $f=\sum a_n z^n$ and $g=\sum b_n z^n$ have only isolated singularities and (1) $h=\sum a_n b_n z^n$. The restriction to isolated singularities (and to single sheets of Riemann surfaces) is then dropped, and the theorem is formulated and proved along lines which were suggested by Borel in a remark which simultaneously warned that this approach would lead to complicated statements. The theorem in its general form asserts that if α and β are the sets of nonregular points of f and g, respectively, then each singular point of h lies either at one of the points of the set $\alpha\beta$ or at the origin. Here $\alpha\beta$ denotes the set of points z_1z_2 with z_1 in α and z_2 in β; a nonregular point of f is a point which lies outside of the (possibly many-sheeted) domain of existence of f; a singular point is a point on the boundary of this domain. The point $z=0$ on that sheet of the Riemann surface of h which is associated with the element $\sum a_n b_n z^n$ is of course regular; that the points $z=0$ on other sheets need not be regular is shown by the example $f=g=\sum z^n/n$, $h=-\int_0^z t^{-1}\log(1-t)dt$.

Hurwitz combined the functions $f=\sum a_n/z^{n+1}$ and $g=\sum b_n/z^{n+1}$ by the "symbolic addition"

$$h=\sum[a+b]_n/z^{n+1}, \tag{2}$$

where

$$[a+b]_n=\sum_{\nu=0}^{n}\binom{n}{\nu}a_{n-\nu}b_\nu.$$

Here the singularities of h lie in the set $\alpha+\beta$, that is, in the set of points z_1+z_2 with z_1 in α and z_2 in β. The author generalizes further:

First, by dealing with methods of combining f and g in such a way that the singularities of h lie in a set γ of points $r(z_1, z_2)$ (z_1 in α, z_2 in β), where $r(z, t)$ is continuous in z and t and analytic in each of the variables when the other is kept constant.

Second, by admitting several functions $f_a, f_b, \cdots, f_p$ with series $\sum a_n z^n, \cdots, \sum p_n z^n$ and replacing (1) by the general formula,

$$h(z)=\sum P(a_n, b_n, \cdots, p_n)z^n,$$

where P denotes a polynomial. Here the singularities of h lie in the set $P(\alpha, \beta, \cdots, \pi)$, where the symbols $\alpha, \beta, \cdots, \pi$ run through all singularities of $f_a, f_b, \cdots, f_p$; where the power α^k is to be interpreted as a product of k factors in α, not necessarily identical; and where addition is in the set-theoretical sense.

Third, by starting with functions $f_a=\sum a_n/z^{n+1}, \cdots, f_p=\sum p_n/z^{n+1}$ and using a combination of the form

$$h=\sum[P(a, b, \cdots, p)]_n/z^{n+1},$$

where the coefficients $[P]_n$ are defined by a formula analogous to that under (2). Here the singularities of h lie in the set $P(\alpha, \beta, \cdots, p)$, where products and powers are to be interpreted as above, and addition is in the arithmetical sense.

The paper includes extensive discussions of the literature on its subject, and it is liberally illustrated with examples.
G. Piranian (Ann Arbor, Mich.).

✱Tammi, Olli. On the coefficients of bounded schlicht functions. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 297–301 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author summarizes results he has recently obtained and already published in full detail [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. nos. **114** (1952); **149**, 162 (1953); these Rev. **14**, 366; **15**, 302, 516]. He announces a general conjecture on the form the image domain takes for the function maximizing $|a_n|$ for the class of functions bounded and univalent in $|z|<1$. *A. W. Goodman.*

✱Kjellberg, Bo. A relation between the maximum and minimum modulus of a class of entire functions. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 135–138 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Let

$$f(z)=\prod_{n=1}^{\infty}\left(1+\frac{z}{r_n}\right), \quad r_n>0,$$

be an integral function of order ρ less than 1, with negative zeros. The author shows very simply that if

$$\liminf_{n\to\infty}\frac{\log f(n)}{r^\lambda}<\infty,$$

where $0<\lambda<\rho$, then, given $\epsilon>0$, we have for some arbitrarily large positive r,

$$\log|f(-r)|-\cos\pi\lambda\log|f(r)|>r^{\rho-\epsilon}.$$

This result is particularly striking since, if $\rho=\lambda$, (1) is insufficient to imply (2), as the function $f(z)=\cosh\sqrt{z}$ shows, for which $|f(-r)|\leqq 1$, while $\lim_{r\to\infty} r^{-1/2}\log f(r)=1$.
W. K. Hayman (Exeter).

Kumar Jain, Mahendra. On the derivatives of integral functions. Ganita **4**, 143–146 (1953).

Let $f(z)$ be an entire function of finite order ρ and lower order λ; $A(r)$, $A^{(1)}(r)$ the maximum real parts of f and f', respectively; $\mu(r)$, $\mu^{(1)}(r)$, the maximum terms in the power series of f and f'. Then, in each case for sufficiently large r, if $\rho<\frac{1}{2}$, $A(r)>A^{(1)}(r)$; if $\lambda>1$, $A(r)<A^{(1)}(r)$; if $\rho<1$, $\mu(r)>\mu^{(1)}(r)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Korevaar, Jacob. Entire functions as limits of polynomials. Duke Math. J. **21**, 533–548 (1954).

Let R be a set of points in the z-plane, $C(R)$ the class of the, clearly entire, functions $f(z)\not\equiv 0$ for which there exist polynomials $f_n(z)$ that tend to $f(z)$ as $n\to\infty$ uniformly in any bounded domain and have all their zeros in R. If $C(R)$ consists of all entire $f(z)\not\equiv 0$ whose zeros lie in R, R is said to be regular. The cases when R is a half-line or a line were dealt with by Laguerre, further cases by Pólya, Obrechkoff [cf. Obrechkoff, Quelques classes de fonctions entières . . . , Hermann, Paris, 1941; these Rev. **7**, 516], and by the author [same J. **18**, 573–592 (1951); these Rev. **13**, 222].

A considerable part of the author's work was based on geometrical properties of R; and several criteria (§§5, 6) were given for a set to be regular. In the present paper surprisingly plain results of the following type are established: R is regular if and only if $C(R)$ contains entire functions of arbitrarily large finite order, or a function of infinite order. Hence a non-regular R has a finite "order" $\omega(R)$, the least upper bound of the orders ρ_f of the $f(z)\in C(R)$. If $f(z)$, of order $\rho\geqq 0$, has its zeros in R, while it does not belong to $C(R)$, then $\omega(R)\leqq 2[\rho]$, and this estimate is best possible. There are further theorems, for instance: if $\exp(\sum_1^q a_j z^j)\in C(R)$, then $\exp(\sum_1^q \lambda a_j z^j)\in C(R)$ for any $\lambda\geqq 0$. The following, however, is the key result (cf. Theorem 5.1). Let $C(R)$ contain an $f(z)$ of order $\tau>1$ and let r be the greatest integer $<\frac{1}{2}\tau$. Then there is a complex number $\Omega\neq 0$ such that, whenever b_{r+1}/Ω is real,

$$\exp(b_1z+\cdots+b_rz^r+b_{r+1}z^{r+1})\in C(R).$$

The proofs are based on the discussion of the distribution of the zeros $z_{n\nu}$ of a sequence $\{f_n(z)\}$ of approximating polynomials (§3) and on deriving new sequences from a given one (§4) by selecting suitable subsequences of the zeros.
H. Kober (Birmingham).

Wilson, R. Determinantal criteria for meromorphic functions. Proc. London Math. Soc. (3) **4**, 357–374 (1954).

Let the function $f(z)=\sum c_n z^n$ be regular in $|z|<1$; let its only singularities on $|z|=1$ be l_1 poles of order m_1, l_2 poles of order $m_2, \cdots$ $(m_1>m_2>\cdots; \sum l_i m_i=q)$, and let $D_{n\nu}$ denote the determinant of order ν whose element in the kth row and pth column is c_{n+k+p}. Then it follows from Hadamard's work [J. Math. Pures Appl. (4)**8**, 101–186 (1892)] that

$$\limsup|D_{n\nu}|^{1/n}=1 \quad\text{for}\quad \nu=1, 2, \cdots, q-1,$$
$$\lim|D_{nq}|^{1/n}=1,$$
$$\limsup|D_{n\nu}|^{1/n}<1 \quad\text{for}\quad \nu>q.$$

The author launches a more delicate investigation concerning the quantities

$$(1)\qquad \lambda_\nu=\limsup_{n\to\infty}\frac{\log|D_{n\nu}|}{\log n}\qquad(\nu=1, 2, \cdots, q).$$

The points (ν, λ_ν), with λ_0 defined to be zero, determine a convex polygonal line in the Cartesian plane. This polygonal line, called the order diagram of the function, is sym-

metric with respect to the perpendicular bisector of the segment $[0, q]$ on the horizontal axis. Its vertices correspond to those values of ν for which the superior limit in (1) is a proper limit.

The first vertex of the order diagram, not counting the origin, is at the point $[l_1, l_1(m_1-1)]$. The second vertex is at the point

$$\begin{array}{ll} [l_1+l_2, l_1(m_1-1)+l_2(m_2-1)] & \text{if } m_2=m_1-1, \\ [2l_1+l_2, 2l_1(m_1-2)+l_2(m_2-1)] & \text{if } m_2=m_1-2, \\ [2l_1, 2l_1(m_1-2)] & \text{if } m_2<m_1-2. \end{array}$$

G. Piranian (Ann Arbor, Mich.).

Meschkowski, Herbert. Beiträge zur Theorie der Orthonormalsysteme. Math. Ann. **127**, 107–129 (1954).

Let B be a multiply-connected domain in the complex plane with an analytic boundary C; let Ω be the class of all functions $\varphi(z)$ which are regular analytic in B, have a finite Dirichlet integral there and possess a single-valued integral function $\phi(z)$ in B. The scalar product $(f, \varphi)=\iint_B f\bar{\varphi}dxdy$ of two functions in Ω can also be written as $(2i)^{-1}\int_C f\bar{\phi}dz$ provided that f and ϕ are continuous in $B+C$. This latter form permits an extension of the definition to the case that $f(z)$ is meromorphic in B. The author sets himself the problem to express meromorphic functions in B in terms of a complete orthonormal system $\{\varphi_\nu\}$ in Ω and of a suitable set of meromorphic standard functions. He introduces functions $N_n'(z, u)$ which are analytic in z, except for a pole of order $n+1$ at the parameter point u, and which are orthogonal to all functions in Ω. Given a meromorphic function $f(z)$ we may subtract a proper linear combination of singularity functions $N_n'(z, u)$ and obtain a remainder term in Ω which has the same Fourier coefficients with respect to the $\{\varphi_\nu\}$ as $f(z)$. The functions $N_n'(z, u)$ can be expressed in terms of well-known domain functions and canonical mappings. For example, let $E(z, u)$ be that univalent function which maps B onto a circle slit along concentric circular arcs such that $z=u$ goes into the center; then

$$N_0'(z, u)=-\partial \log E(z, u)/\partial u$$

and all other $N_n'(z, u)$ can be obtained by repeated differentiation with respect to the parameter point u. The author considers further functions $\{\chi_\nu(z)\}$ which are regular in $B+C$ and orthonormalized in the Szegö metric based on the scalar product $[f, g]=\int_C f\bar{g}ds$. He obtains analogous singularity functions and representation theorems. He discusses the extension of his results to the case that the meromorphic function $f(z)$ is not continuous on the boundary C. In the rest of the paper the author discusses various possible choices and definitions of orthonormal systems. He shows, for example, that a system of determinants of well-known domain functions forms a complete orthonormal set in Ω and is identical with a function system defined by Bergman by use of a set of extremum problems. Finally, he shows that if B has a certain canonical form the kernel function can be obtained by integration of a rational function; he determines the corresponding canonical mapping for the case of a circular annulus. *M. Schiffer* (Stanford, Calif.).

Leja, F. Polynômes extrémaux et la représentation conforme des domaines doublement connexes. Ann. Polon. Math. **1**, 13–28 (1954).

The author generalizes the Tchebycheff polynomials to doubly-connected domains and uses them to construct the mapping onto a circular annulus. *H. L. Royden.*

Walsh, J. L., und Gaier, D. Zur Methode der variablen Gebiete bei der Randverzerrung. Arch. Math. **6**, 77–86 (1954).

La méthode des domaines variables, dont le principe remonte à Montel-Carathéodory, est appliquée ici à la limitation de l'oscillation de arg $(f(z)-1)(z-1)^{-1}$, où $f(z)$ représente conformément le demi-plan $\Re z<1$ sur un domaine G admettant $w=1$ pour point frontière accessible correspondant à $z=1$. Un exemple prouve ensuite que la double inégalité obtenue est la meilleure possible. L'article se termine par une comparaison entre la propriété de Visser $[f'(z)(f(z)-1)^{-1}(z-1)\to 1$ angulairement$]$ et la semi-conformité d'Ostrowski, qui apparait comme plus précise.

Note du référent. La double inégalité établie au début de cet article s'applique à un cas plus général, en couvenant de définir les "Grenzstützen" extérieurs de G par des suites de points de densité 1 au point $w=1$. D'autre part la démonstration semble pouvoir être simplifiée par application du principe de la mesure harmonique, ainsi qu'il a été indiqué par le Ref. pour une inégalité analogue [C. R. Acad. Sci. Paris **212**, 977–980 (1941); ces Rev. **5**, 37]. Signalons enfin que la propriété de Visser admet un critère géométrique, car elle équivaut à la convergence vers la bande $|\Im\zeta|<\frac{1}{2}\pi$, quand $a\to 1$, sur un arc d'acces convenable de l'image Ω_a de G par $\zeta=\log(w-1)-\log(a-1)$. *J. Lelong-Ferrand.*

Ahlfors, Lars V. On quasiconformal mappings. J. Analyse Math. **3**, 1–58; correction, 207–208 (1954).

Let $\zeta=\zeta(z)$ be a differentiable homeomorphism of the z-plane to the ζ-plane. Setting $p=\partial\zeta/\partial z$ and $q=\partial\zeta/\partial\bar{z}$, we call the quantity $(|p|+|q|)(|p|-|q|)^{-1}$ the dilation of the mapping ζ. A differentiable mapping with a bounded dilation is called quasiconformal. This concept extends immediately to mappings of one Riemann surface onto another.

O. Teichmüller conjectured that, in a given homotopy class of mappings of one compact Riemann surface W onto another W', the homeomorphism which has the smallest maximal dilation is either a conformal mapping or else satisfies

$$(*)\qquad \frac{q}{p}=k\frac{f}{|f|}$$

for a positive constant $k<1$ and a quadratic differential fdz^2 on W. If W and W' have boundary contours, the same result holds under a suitable definition of homotopy with f real along the boundary contours. If prescribed points of W are required to go into prescribed points of W', then fdz^2 may have simple poles at these points.

The present paper is primarily devoted to giving a proof of the Teichmüller conjectures. The author generalizes the concept of quasiconformality to homeomorphisms which are not necessarily differentiable by saying that a homeomorphism has dilation at most K if each topological rectangle in the Z-plane is mapped into a rectangle in the ζ-plane whose conformal modulus is at most K times the conformal modulus of the original rectangle. This definition agrees with the earlier one for differentiable mappings and has the advantage that under certain trivial conditions a family of such mappings with a uniformly bounded dilation is a normal family and the limit mapping also has a bounded dilation.

The proof of Teichmüller's conjecture consists of two parts. The first step is to show that any mapping satisfying (*) has a smaller maximal dilation than any other homeomorphism in the same homotopy class. The proof uses the length-area principle and is due to Teichmüller except for

taking care of the generality of using non-differentiable mappings.

The second and more difficult step is to prove the existence of a homeomorphism satisfying (*) in each homotopy class of homeomorphisms. This Ahlfors does by introducing mean quasiconformal mappings, i.e., mappings for which the integral of the mth power of the dilation is finite, and minimizing the mean dilation. Conditions are found which the extremal mappings must satisfy, and it is shown that as $m\to\infty$ these extremal mappings tend to a mapping which satisfies (*).

Let T_g be the space of all Riemann surfaces of genus g with suitable topological fixing. Let T_g be metrized by defining the distance between two surfaces as the log of the smallest maximal dilation of homeomorphisms in the proper homotopy class of mappings between them. Then T_g becomes a metric space and it is a consequence of the Teichmüller conjecture that T_g is homeomorphic to the $(6g-6)$-dimensional Euclidean space. *H. L. Royden.*

Gerstenhaber, Murray, and Rauch, H. E. On extremal quasi-conformal mappings. I. Proc. Nat. Acad. Sci. U. S. A. **40**, 808–812 (1954).

Let f be a quasi-conformal mapping of a Riemann surface S_1 onto a Riemann surface S_2, and let $\eta=\lambda dw d\bar{w}$ be a conformal metric on S_2. If both η and f are assumed suitably differentiable, then η induces a metric

$$f^*\eta=a(z)dz^2+\bar{a}(z)d\bar{z}^2+b(z)dzd\bar{z}$$

on the surface S_1. The authors show that the condition $a(z)$ analytic is a necessary condition that for a given η the function f minimizes the Douglis-Dirichlet integral $I(f^*\eta)=\frac{1}{2}i\int\int b\,dzd\bar{z}$. The authors plan to use this fact as a starting point for the solution of the Teichmüller problem [cf. the paper reviewed above] by techniques from the theory of minimal surfaces. *H. L. Royden.*

Gerstenhaber, Murray, and Rauch, H. E. On extremal quasi-conformal mappings. II. Proc. Nat. Acad. Sci. U. S. A. **40**, 991–994 (1954).

In the present paper the authors complete their task of showing formally that the extremal quasi-conformal mapping of a Riemann surface S_1 onto S_2 is given by maximizing over all conformal metrics η on S_2 the minimum of the Douglis-Dirichlet integral $I(f^*\eta)$ over all homeomorphisms f. To make the proof rigorous one must show the existence of a unique minimizing homeomorphism for each η and show the continuous dependence of the homeomorphism on η. *H. L. Royden* (Stanford, Calif.).

Seibert, Peter. Über die bei Deformationen Riemannscher Flächen mit endlich vielen Windungssorten entstehenden Randstellen. Arch. Math. **5**, 389–400 (1954).

The author gives, with the aid of the Speiser-Elfving "Streckenkompexe", a method for the modification of a simply-connected covering surface of the type studied by R. Nevanlinna which is ramified over a finite number of points of the extended plane. The class of covering surfaces so obtained yields "Randstellen" of all four types in the sense of E. Ullrich [Jber. Deutsch. Math. Verein. **46**, 232–274 (1936)]. *M. Heins* (Providence, R. I.).

Tsuji, Masatsugu. A metrical theorem on the singular set of a linear group of Schottky type. J. Math. Soc. Japan **6**, 115–121 (1954).

Let G be a Schottky group in the plane, E its singular set, and C_1 one of the boundary contours of a fundamental region of the group. Let E_1 be the set of singular points of G which are contained in C_1 and in infinitely many equivalents of C_1. Then E_1 has positive capacity. If the universal covering surface of the complement of E is mapped onto the unit circle, then the image of E_1 has positive measure on the unit circumference. *H. L. Royden.*

Tsuji, Masatsugu. On Neumann's problem for a domain on a closed Riemann surface. J. Math. Soc. Japan **6**, 122–128 (1954).

The Neumann problem for a domain of a compact Riemann surface is treated by the method developed by L. Myrberg for the case of the interior of the unit circle [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **103** (1951); these Rev. **13**, 743]. *M. Heins* (Providence, R. I.).

Ohtsuka, Makoto. Boundary components of Riemann surfaces. Nagoya Math. J. **7**, 65–83 (1954).

L'A. poursuit d'une part l'étude de la correspondance entre frontière idéale et cercle-unité, d'autre part celle du problème de Dirichlet pour des données sur cette frontière [voir même J. **3**, 91–137 (1951); ces Rev. **13**, 642]. Le résultat essentiel est l'obtention des conditions de résolubilité pour des valeurs données sur l'espace topologique $\mathfrak{C}$ des éléments-frontière de Kerékjártó [Vorlesungen über Topologie, Springer, Berlin, 1923]. Cependant, comme le remarque l'A., ce problème est assez particulier puisque, dans le cas d'un domaine simplement connexe, il n'y a qu'un seul élément-frontière, donc qu'une seule valeur donnée, la solution étant une constante. Il serait désirable d'étendre le problème à des éléments-frontière plus fins, par exemple ceux de R. S. Martin [voir M. Parreau, Ann. Inst. Fourier Grenoble **3**, 103–197 (1952); ces Rev. **14**, 263].

I. Soit $\mathfrak{R}$ une surface de Riemann d'ordre de connexion 3 au moins, $f(z)$ la fonction qui représente conformément le cercle $|z|<1$ sur le recouvrement universel de $\mathfrak{R}$, $\mathfrak{G}$ le groupe fuchsien ou fuchsoïde correspondant. Un point de Γ $(|z|=1)$ est dit singulier s'il est limite d'images d'un même point de $\mathfrak{R}$; les points de Γ non singuliers constituent des arcs ouverts dits réguliers; si les extrémités sont des points fixes de transformations de $\mathfrak{G}$, l'arc est dit complètement régulier. Ces points et arcs se répartissent en classes d'équivalence vis-à-vis de $\mathfrak{G}$.

Adjoignons à $\mathfrak{R}$ l'ensemble $\mathfrak{C}$ de ses éléments-frontières de façon à constituer un espace topologique $\mathfrak{R}^*=\mathfrak{R}\cup\mathfrak{C}$. A chaque point Q de $\mathfrak{C}$ correspond d'une manière facile à définir un ensemble de telles classes, soit $\psi(Q)$. Si $Q\neq Q'$, $\psi(Q)$ et $\psi(Q')$ sont disjoints. Supposons qu'un point isolé Q de $\mathfrak{C}$ ait un voisinage V dans $\mathfrak{R}^*$ homéomorphe à un disque. Selon que $V-\{Q\}$ est conformément équivalent à un disque pointé ou à une couronne, on dit que Q est un élément-frontière parabolique, ou "hyperbolique isolé de 1re classe" (IH1). L'A. démontre les résultats suivants: 1. Si Q est parabolique (ou IH1), $\psi(Q)$ est une classe unique de points fixes paraboliques de $\mathfrak{G}$ (ou d'arcs complètement réguliers), et inversement à chacune de ces classes correspond un point Q parabolique (ou IH1). 2. Si Q n'appartient pas aux deux espèces précédentes, $\psi(Q)$ se compose d'un ensemble non dénombrable de points singuliers non-fixes, et éventuellement d'un ensemble dénombrable de classes d'arcs non-complètement réguliers. 3. Un chemin s'en allant à l'infini sur $\mathfrak{R}$ tend vers un point Q de $\mathfrak{C}$; si deux chemins tendant vers un même point Q sont homotopes sur $\mathfrak{R}$, ils admettent deux images qui tendent vers un même point ou arc de Γ, et réciproquement. 4. Soit $\mathfrak{R}_n$ une suite croissante de surfaces dont la réunion est $\mathfrak{R}$, et telles que $\overline{\mathfrak{R}}_n$ soit compacte

dans $\mathfrak{R}$. L'image de $\mathfrak{R}_n$ dans $|z|<1$ peut comprendre plusieurs composantes; l'ensemble A_n des points non fixes de Γ qui sont sur la frontière d'une telle composante, sans être contenus dans la réunion C des $\psi(Q)$, ont la puissance du continu pour chaque n. Si et seulement si $\mathfrak{R}$ est de connexion infinie, il y a des points non fixes, dont l'ensemble a la puissance du continu, qui n'appartiennent ni à C ni à aucun A_n.

II. La méthode des enveloppes de Perron-Brelot [voir par ex. M. Brelot, Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 22, 167–200 (1946); ces Rev. 8, 581] s'applique à $\mathfrak{R}^*$. Etant donnée une fonction $\varphi(Q)$ sur $\mathfrak{C}$, elle conduit à deux fonctions $\underline{H}_\varphi(P)$ et $\bar{H}_\varphi(P)$. Si elles sont égales, leur valeur commune $H_\varphi(P)$ est harmonique et φ est dite résolutive. On suppose que $\mathfrak{R}$ a une frontière idéale positive. L'A. démontre les propositions suivantes: 1. La fonction caractéristique φ_G d'un ensemble ouvert G de $\mathfrak{C}$ est résolutive. 2. P étant fixé dans $\mathfrak{R}$, il existe une mesure de Radon sur $\mathfrak{C}$ et une seule qui, pour chaque ouvert G de $\mathfrak{C}$, prend la valeur $\mu^P(G)=H_{\varphi_G}(P)$. Les ensembles mesurables par rapport à μ^P ne dépendent pas de P. 3. On a

$$\underline{H}_\varphi(P)=\underline{\int_{\mathfrak{C}}}\varphi d\mu^P,\quad \bar{H}_\varphi(P)=\overline{\int_{\mathfrak{C}}}\varphi d\mu^P.$$

4. Pour que φ soit résolutive, il faut et il suffit qu'elle soit intégrable et d'intégrale finie par rapport à μ^P; cette condition ne dépend pas de P. 5. L'ensemble C de Γ a pour mesure 2π. Si on transporte φ sur C, la méthode de Perron-Brelot appliquée à $|z|\leqq 1$ conduit à des résultats dont on déduit les précédentes par $P=f(z)$. Un ensemble de $\mathfrak{C}$ est mesurable par rapport à μ^P si et seulement si son image sur Γ est mesurable. *R. de Possel* (Alger).

Hitotumatu, Sin, and Kôta, Osamu. Ideals of meromorphic functions of several complex variables. Math. Ann. 125, 119–128 (1952).

The authors extend the theory of integral ideals of holomorphic functions developed by H. Cartan and K. Oka to fractional ideals. A non-empty set $\mathfrak{a}$ of meromorphic functions in a domain G of the space C^n of n complex variables is called a fractional ideal if the following conditions are fulfilled. a) $\mathfrak{a}$ is a module over the ring of the holomorphic functions in G. b) There exists a non-vanishing meromorphic function φ in G, such that all functions in $\varphi\cdot\mathfrak{a}$ are holomorphic in G (φ is called an "integralizator" of $\mathfrak{a}$).

It is shown, that the main theorems of the theory of integral ideals mutatis mutandis are valid for fractional ideals too. For example, the authors prove (after having introduced the notion of a closed ideal in an obvious manner): If $\mathfrak{a}$ is a closed ideal in a univalent and bounded domain ϑ of holomorphy, then every meromorphic function ψ in ϑ, the restriction of which to every point $P\in\vartheta$ belongs to that ideal $(\mathfrak{a})_P$ which is generated in P by $\mathfrak{a}$, is an element of $\mathfrak{a}$.

Finally, the authors apply their theorems to questions concerning the characterisation of the "ideal class group". *R. Remmert* (Münster).

Popov, B. S. Sur les fonctions paraanalytiques à deux dimensions. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire 6 (1953), no. 1, 29 pp. (1954). (Macedonian. French summary)

The author shows that p. (paraanalytic) functions of classes D, P [M. Fréchet, C. R. Acad. Sci. Paris 235, 1585–1587 (1952); 236, 348–351 (1953); these Rev. 14, 463, 865] have properties such as the following. For the class P: the transformation by a p. function preserves the quotient of the angular coefficients of the tangents. For the class D: the transformation by a p. function preserves the difference of the above coefficients. The notion of integral can be introduced for these classes, as well as theorems analogous to those given by Cauchy for analytic functions.

W. J. Trjitzinsky (Urbana, Ill.).

Takasu, Tsurusaburo. A complex function theory on a "supra-corpus" of n-dimensional hypercomplex numbers. I. Yokohama Math. J. 1, 131–224 (1953).

An extended treatment of the function theory of a commutative hypercomplex variable w, where w is an arbitrary element of a linear algebra over the complex field having finite order n and a unity element. Analytic functions $f(w)$ are defined in accordance with Scheffers' original definition of differentiability, namely: there exists an element of the algebra f' such that $df=f'dw$ for all dw. [The reviewer calls attention to the algebraic definition recently given by J. A. Ward, Proc. Amer. Math. Soc. 4, 456–461 (1953); these Rev. 15, 6.] The larger class of polygenic functions [first studied for a general hypercomplex variable by Ketchum and Martin, Bull. Amer. Math. Soc. 38, 66–72 (1932)], in which df but not necessarily f' exists, is also considered. In the algebraic preliminaries the importance of the rôle of nilfactors as generalizations of the zero element is properly stressed. Representations in terms of idempotent and nilpotent units are utilized in a manner similar to decompositions into subalgebras which were used earlier by Ketchum [Trans. Amer. Math. Soc. 30, 641–667 (1928)]. Various known analogues of theorems in ordinary complex variable theory such as the Cauchy-Riemann equations and Cauchy's integral theorem are obtained, as well as certain analogues not previously discussed such as conjugate hypercomplex numbers, stereographic projection, infinity factors (generalization of the point at infinity), Abel's limiting-value theorem, and the Beltrami-Pompéiu integral formula. The properties of some elementary functions are discussed. Also, conditions for the vanishing of certain multiple hypercomplex integrals are found by use of a generalized Stokes' theorem. [However, these conditions are not related to differentiability as in Cauchy's theorem. Moreover, the author's definition of multiple hypercomplex integrals is not intrinsic because the value of the integral depends on the representation of the algebra.] The detailed discussion of convergence and integration differs from earlier treatments in that an "absolute value" with n components is used. This absolute value, $A(w)$, seems to the reviewer to be equivalent to a metric which is the maximum of the absolute values of the components in the idempotent directions. This, in turn, would be equivalent to the ordinary Euclidean metric, $E(w)$, previously used by the reviewer, provided there are no nilpotent numbers. If there are nilpotent numbers, then $A(w)$ would be improper in the sense that it could vanish for a non-zero w, and limits would not be unique. Theorems which involve the value of a limit, such as those about manipulations with series or integrals, would only hold to within an arbitrary additive nilpotent number. Thus the theorems obtained by use of $A(w)$ appear to be essentially weaker than previous results found with the aid of $E(w)$, and are, in fact, what one would get from $E(w)$ by considering the algebra modulo the nilpotent numbers. The author's absolute value may be appropriate for the study of the convergence of power series, and has in effect been so used by the reviewer, but the argument in favor of

its use for determining the topology of the space is not convincing. *P. W. Ketchum* (Urbana, Ill.).

Yoneda, Keizo. An integration theory in the general bicomplex function theory. Yokohama Math. J. 1, 225–262 (1953).

This paper covers in more detail part of the same material as in the article reviewed above for the special case of a class of hypercomplex variables of order 4. No use is made of idempotent and nilpotent representations. An absolute value is used which appears to be even weaker than Takasu's and amounts to considering the algebra modulo all nilfactors. Conditions for the vanishing of certain double and triple hypercomplex integrals are worked out in detail, but are subject to the same limitations mentioned in the preceding review. Also, some of these conditions are incorrect, at least in special cases where there are nilpotent numbers, because the possibility of the vanishing of the multiplication parameters has been ignored. *P. W. Ketchum* (Urbana, Ill.).

Theory of Series

Kangro, G. Summability factors for the method of weighted arithmetic means. Doklady Akad. Nauk SSSR (N.S.) 99, 9–11 (1954). (Russian)

Necessary and sufficient conditions for the ϵ_n are given in order that a B-summable series $\sum \epsilon_n u_n$ should correspond to each P-summable series $\sum u_n$ (or a B-summable series to each $|P|$-summable, or a $|B|$-summable to each $|P|$-summable, or a B-summable to each P-bounded series). Here B is an arbitrary method and P a method of weighted arithmetic means (equivalently, a Riesz $R(\lambda_n, 1)$ method with $\Delta\lambda_n \neq 0$). The author's conditions may be derived by an application of known theorems (by Toeplitz, Schur, Hahn and Mears) which characterize regular, absolute regular, and some other types of matrix methods.

G. G. Lorentz (Detroit, Mich.).

Martin, C. F. A note on a recent result in summability theory. Proc. Amer. Math. Soc. 5, 863–865 (1954).

In a recent paper A. M. Tropper [same Proc. 4, 671–677 (1953); these Rev. 15, 118] proves the following theorem. In order that the regular normal matrix A shall sum a bounded divergent sequence, it is sufficient that (a) its unique reciprocal B shall not be regular and (b) there exists a normal matrix Q with $\|Q\| < \infty$ whose columns are all null sequences, such that the matrix $C = BQ$ has bounded columns and $\|C\| = \infty$. The necessity of condition (a) was pointed out. In the present paper the author proves that condition (b) is necessary also. *V. F. Cowling.*

Korevaar, Jacob. Kloosterman's method in Tauberian theorems for C_k summability. Proc. Amer. Math. Soc. 5, 574–577 (1954).

H. D. Kloosterman used a mean-value theorem connecting $\Delta_k^r f(x)$, $f^{(r)}(x)$ and $f^{(r+1)}(\xi)$ to obtain Tauberian theorems [J. London Math. Soc. 15, 91–96 (1940); these Rev. 2, 89]. The reviewer developed Kloosterman's ideas further, replacing his mean-value theorem by an identity, and extending the results to fractional orders [ibid. 18, 239–248 (1943); these Rev. 6, 42]. Kloosterman later gave a further mean-value theorem connecting $\Delta_h^{r+\nu} f(x)$ $(\nu = 0, 1, \cdots, k-1)$, $f^{(r)}(x)$ and $f^{(r+k)}(\xi)$ [Duke Math. J. 17, 169–186 (1950); these Rev. 11, 716]. Here the author states a variant of the last formula, and shows that it may be applied in just the same way as the formulae of Kloosterman and the reviewer. The author states that his own formula may be obtained by arguments of Markoff, and points out that certain of Kloosterman's results were anticipated by Markoff [The calculus of finite differences, v. I, II, St. Petersburg, 1891; German translation published by Teubner, Leipzig, 1896]. The results in question seem to be the formulae attributed to Markoff in Milne-Thomson's "The calculus of finite differences", §7.02 [Macmillan, London, 1933].

L. S. Bosanquet (London).

Pati, T. On the second theorem of consistency in the theory of absolute summability. Quart. J. Math., Oxford Ser. (2) 5, 161–168 (1954).

Chandrasekharan [J. Indian Math. Soc. (N.S.) 6, 168–180 (1942); these Rev. 5, 63] proved a theorem for absolute summability of Dirichlet series by Riesz means analogous to Hardy's [Proc. London Math. Soc. (2) 15, 72–88 (1916)] well known "second theorem of consistency" for Riesz summability. In the present paper the author extends a theorem of the type proven by Chandrasekharan. A typical theorem is the following: If $\phi(t)$ is a non-negative and monotonic increasing function of t for $t \geqq 0$ steadily tending to infinity as t tends to infinity, such that, for positive integral k, $\phi(t)$ is a $(k+1)$th indefinite integral for $t \geqq 0$, and $t^r\phi^{(r)}(t)/\phi(t)$ is of bounded variation on the interval (h, ∞) for $(r = 1, 2, \cdots, k)$, where h is a finite positive number, then any infinite series which is summable $|R, \lambda_n, k|$ is also summable $|R, \phi(\lambda_n), k|$. *V. F. Cowling.*

Jurkat, Wolfgang B. Questions of signs in power series. Proc. Amer. Math. Soc. 5, 964–970 (1954).

In der Arbeit werden Aussagen gemacht über die Vorzeichen von k_n in der formalen Relation

$$\sum k_n x^n = (\sum q_n x^n)/(\sum p_n x^n);$$

dieses Problem ist im Sonderfall $q(x) \equiv 1$ bedeutsam für Vergleichssätze bei Nörlund-Verfahren [vgl. G. H. Hardy, Divergent series, Oxford, 1949, §4.5; diese Rev. 11, 25]. Satz 1 (bzw. Satz 2) lautet: Aus $p_n > 0$, $p_{n+1}/p_n \uparrow$ $(0 \leqq n \uparrow)$ und $q_n/p_n \uparrow$ (bzw. $p_n/q_n \uparrow$) $(0 \leqq n \uparrow)$ folgt stets $k_n \geqq 0$ (bzw. $k_n \leqq 0$) für $n \geqq 1$. Dies verallgemeinert Ergebnisse von Szegö [Math. Z. 25, 172–187 (1926)] und Kaluza [ibid. 28, 161–170 (1928)], die den Sonderfall $q(x) \equiv 1$ betrachteten. Dieselbe Aussage folgt unter den veränderten Voraussetzungen $p_0 > 0$, $p_n \downarrow$ $(0 \leqq n \uparrow)$ und $p_0(q_n - q_{n-1}) \geqq q_0(p_n - p_{n-1})$ (bzw. $p_0(q_n - q_{n-1}) \leqq q_0(p_n - p_{n-1})$) für $n \geqq 1$ (Satz 4–5), wofür zwei Beweise gegeben werden. Die Voraussetzungen zu Satz 1 und 4 werden verglichen. In den Sätzen 8–11 beziehen sich allgemeiner die Annahmen und die Aussagen auf die α-ten Differenzen von q_n, p_n, k_n. Der Verf. kündigt Anwendungen auf Vergleichssätze bei Nörlund-Verfahren an.

D. Gaier (Stuttgart).

Jurkat, Wolfgang, und Peyerimhoff, Alexander. Lokalisation bei absoluter Cesàro-Summierbarkeit von Potenzreihen und trigonometrischen Reihen. I. Math. Z. 60, 255–270 (1954).

H. C. Chow obtained necessary and sufficient conditions for a power series $\sum a_n z^n$ to be summable $|C, \alpha|$, $\alpha > 0$, at $z = 1$ in the form:

$$(*) \qquad \sum n^{-\alpha} |a_n| < \infty$$

together with a suitable "local condition". The local condition is satisfied in particular if the sum-function is regular

at $s=1$ [Chow, Quart. J. Math., Oxford Ser. (2) 4, 152–160 (1953); these Rev. 15, 26]. Independently of Chow the authors obtained the case $\alpha=0$ of the following theorem, proved here for $\alpha>-1$: the power series is summable $|C, \alpha|$ at $s=1$ if

$$(^{**})\qquad \sum|\Delta(n^{-\alpha}a_n)|<\infty$$

and the sum-function is regular at $s=1$; Δ may be replaced by Δ^p for some integer p [Jurkat and Peyerimhoff, Arch. Math. 4, 285–297 (1953); these Rev. 15, 617; cf. the theorems of Fatou and Riesz: M. Riesz, J. Reine Angew. Math. 140, 89–99 (1911)]. They also give "localization theorems" (obtained previously for $\alpha=0$) for summability $|C, \alpha|$, $\alpha>-1$, of trigonometrical series with coefficients satisfying (**) [Jurkat and Peyerimhoff, loc. cit.; cf. the theorems of Riemann and Zygmund: Zygmund, Math. Z. 24, 47–104 (1925)]. *L. S. Bosanquet* (London).

Moore, Charles N. On relationships between Nörlund means for double series. Proc. Amer. Math. Soc. 5, 957–963 (1954).

Certain theorems concerning relationships between Nörlund means for single series are generalized to double series. If p_{ij}, q_{ij} are two infinite sequences of non-negative numbers, the author writes

$$P_{mn}=\sum_{i=0,\, j=0}^{m,n} p_{ij},\quad Q_{mn}=\sum_{i=0,\, j=0}^{m,n} q_{ij},$$

$$N_{mn}^{(p)}(s)=\sum_{i=0,\, j=0}^{m,n} p_{m-i,\, n-j}s_{ij}/P_{mn}$$

(and p, P replaced by q, Q),

where the s_{ij} are the partial sums of a double seres $\sum u_{ij}$, which remain bounded for all (i, j). Necessary and sufficient conditions for the regularity of Nörlund means for such double series have been given by the author in his book "Summable series and convergence factors" [Amer. Math. Soc. Colloq. Publ., v. 22, New York, 1938, Ch. II, Th. II]. It is now shown that any two regular Nörlund methods for summing double series where each row and each column furnishes a regular method for simple series are consistent. Next, if (N, p_{mn}) and (N, q_{mn}) are two such regular Nörlund methods, necessary and sufficient conditions are found in order that (N, q_{mn}) should include (N, p_{mn}). Finally, necessary and sufficient conditions are obtained in order that (N, p_{mn}) and (N, q_{mn}) should be equivalent (mutually consistent). *R. G. Cooke* (London).

Růžička, Jaroslav. Über die Umordnung unendlicher Reihen von hyperkomplexen Zahlen. Čehoslovack. Mat. Ž. 3(78), 23–73 (1953). (Russian. German summary)

For a sequence x_α of vectors in E^n, let Sx_α (the "sum-region of x_α") be the set of all points $p \in E^n$ such that for some rearrangement y_α of x_α, p is a cluster point of the sequence of partial sums of the series $\sum y_\alpha$. When $\sum x_\alpha$ is convergent, Sx_α is merely the set of all sums of convergent rearrangements of $\sum x_\alpha$. In this case, Steinitz [J. Reine Angew. Math. 143, 128–175 (1913); 144, 1–40 (1914)] showed that Sx_α must be a linear manifold in E^n and described further the relationship between x_α and Sx_α. For the general case, Behrend [Math. Z. 36, 298–301 (1932)] gave a short proof that Sx_α must be a translate of a closed subgroup of E^n, but did not show how the structure of Sx_α is related to that of x_α. In the paper being reviewed, the author describes this relationship under the assumption that x_α is bounded. His methods are generally similar to those of Steinitz [loc. cit.], Wald [Ergebn. Math. Kolloq. 5, 10–13 (1933)], Šklyarskiĭ [Uspehi Matem. Nauk 10, 51–59 (1944); these Rev. 7, 12], and others.

The sequence x_α is called "symmetric" provided for each $z \in E^n$, the positive-term and negative-term subseries of $\sum(z, x_\alpha)$ are either both convergent or both divergent, and it is proved that Sx_α is non-empty if and only if x_α is symmetric. The "convergence-space" of x_α is the set of all $z \in E^n$ for which $\sum(z, x_\alpha)$ is absolutely convergent; this must be a linear subspace of E^n and its orthogonal complement is the "divergence-space" of x_α.

Now for a bounded sequence x_α in E^n, let Mx_α be the smallest closed subgroup of E^n which contains all cluster points of x_α. For each i, let q_i be the point of Mx_α nearest to x_i, and let $\delta_i=x_i-q_i$; δ_α is the "difference-sequence" of x_α. The author's main theorem is as follows: If x_α is a bounded symmetric sequence in E^n, then Sx_α is a translate of the closure of the linear sum $Mx_\alpha+L$, where L is the divergence-space of the difference-sequence of x_α. (In the case treated by Steinitz, this reduces to his result that Sx_α is a translate of the divergence-space of x_α.) *V. L. Klee.*

***Szegö, G. On a theorem of C. Carathéodory.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 62–66. Academic Press Inc., New York, 1954. \$9.00.

A proof of the following theorem of Carathéodory is presented: Given $c_1, \cdots, c_n$ (complex numbers), there exist uniquely determined constants λ_k, ϵ_k $(k=1, \cdots, m;\ m\leqq n)$ such that $\lambda_k>0$, $|\epsilon_k|=1$, $\epsilon_i\neq\epsilon_j$ if $i\neq j$, and

$$c_\nu=\sum_{k=1}^{m}\lambda_k\epsilon_k^{\nu}\quad(\nu=1, \cdots, n).$$

W. Rudin (Rochester, N. Y.).

***van der Corput, J. G. Asymptotic expansions. I. Fundamental theorems of asymptotics.** Department of mathematics, University of California, Berkeley, Calif., 1954. iii+66 pp.

This is the first of a number of self-contained reports on asymptotic expansions. As the author points out, the theory of asymptotic expansions is at present incomplete and has suffered neglect over a long period. This is probably due to the fact that, in practice, asymptotic expansions are largely used in certain kinds of boundary-value problems and the methods established by Poincaré are regarded as satisfactory for this purpose. They have also been used in analytic number theory, in statistical mechanics and in coefficient theory, often in conjunction with the method of steepest descent, but in all these cases conventional methods suffice. Recently extensions of the Poincaré theory have been developed by R. San Juan in dealing with asymptotic expansions arising out of the moment problems associated with different kinds of transform [see, e.g., Revista Mat. Hisp.-Amer. (4) 11, 65–110 (1951); these Rev. 13, 214]. On the other hand, attempts to sharpen the existing theory have also been made in relation to asymptotic expansions near an irregular singularity [see, e.g., Evans, Quart. Appl. Math. 12, 295–300 (1954); these Rev. 16, 131]. Neither of these are referred to in the present work.

The author's aims are both extensive and precise. His intention is to assist mathematicians in the practical use of asymptotic expansions while at the same time building up a rigorous theory on a wide basis. The several reports will cover three successive stages. In the first the Poincaré theory

is subjected to a rigorous analysis. Here a numerical upper bound is not known for the remainder term; all that is known is that its order is the same as that of the first neglected term. For many purposes in applied mathematics this is sufficient. The second stage deals with asymptotic expansions in which the upper bound of the remainder is known. The third stage deals with the transformation of an asymptotic series into one in which the error of approximation may be prescribed.

It is with the first stage that this report is largely concerned. The subjects dealt with are asymptotic equality; asymptotic limits; asymptotically convergent series, single and multiple; uniform asymptotic convergence; analytic functions with prescribed asymptotic expansions; and equations involving asymptotic expansions. The chapter on analytic functions deals with conditions under which a function, analytic in a given region, may have an asymptotic expansion about one or more boundary points of that region. The last chapter concerns solutions in s of the equation $f(s)=t$, where $f(s)$ and t are given asymptotically.

The treatment is logical and rigorous throughout. All the terms used are clearly defined from the start and the basic ideas are subjected to a detailed analysis. There are many examples, carefully chosen to illustrate the points at issue. Although the author does not claim the work to be exhaustive, if the remaining reports maintain the same standard, then the author's aim to provide a rigorous and complete basis for the theory of asymptotic expansions will have been largely achieved. *R. Wilson* (Swansea.)

Slater, L. J. Some new results on equivalent products. Proc. Cambridge Philos. Soc. **50**, 394–403 (1954).

Author's abstract: "In this paper, I prove some new results which are identities connecting groups of general infinite products, and I give two tables, calculated using EDSAC, of the function $1/\prod(1-aq^{n-1})$ over the range $a=-0.90(0.05)+0.95$, $q=0.0(0.05)+1.00$, and of the function $1/\prod(1-q^n)$ over the range $q=0(0.005)+0.890$." Among the results is an equation of modular type, connecting products in q^M and q, with M free parameters. *N. J. Fine* (Philadelphia, Pa.).

Fourier Series and Generalizations, Integral Transforms

***Hyltén-Cavallius, Carl. A positive trigonometrical kernel.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 90–94 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author obtains very concise proofs of some inequalities for trigonometric polynomials $\sum n^{-1}\sin nx$ and generalizations [Turán, Ann. Soc. Polon. Math. **25** (1952), 155–161 (1953); these Rev. **14**, 1080] by using the fact that

$$\sum_{\nu=1}^{\infty}\frac{\sin \nu x}{\nu}\cdot\frac{\sin(\nu-\frac{1}{2})t}{2\sin\frac{1}{2}t}>0$$

for $0<x<\pi$ and all t for which the series converges. *R. P. Boas, Jr.* (Evanston, Ill.).

Tomić, M. Sur les zéros de séries trigonométriques à coefficients monotones. Acad. Serbe Sci. Publ. Inst. Math. **6**, 79–90 (1954).

The author improves several known theorems. The following special cases indicate the character of the results obtained. (I) If $c_k\downarrow 0$, then $\sum_{k=n+1}^{\infty} c_k \sin k\theta$ has at least one zero in each interval $(k-1)\pi/(n+\frac{1}{2})\leqq\theta\leqq k\pi/(n+\frac{1}{2})$, $k=1, 2, \cdots, n$. (II) If $\{c_k\}$ is triply monotonic and $c_k\to 0$, then $\sum_{k=0}^{\infty} c_{n+k+1}\cos(n+2k+1)\theta$ has at least one zero in each interval $k\pi/n\leqq\theta\leqq(k+\frac{1}{2})\pi/(n+1)$, $k=0, 1, 2, \cdots, [n/2]$ (and in the intervals symmetric with respect to $\pi/2$). (III) A theorem on zeros of $\sum_{k=1}^{\infty} c_{n+k+1}\sin(n+2k+1)\theta$ which includes as a special case an inequality of Szegö for the zeros of Legendre polynomials. (IV) If $c_k\downarrow 0$ and $\{kc_k\}$ is strictly convex then the series in (III) has exactly n zeros in $(0, \pi)$. *R. P. Boas, Jr.* (Evanston, Ill.).

Izumi, Shin-ichi. Some trigonometrical series. IX. Tôhoku Math. J. (2) **6**, 30–34 (1954).

If f is continuous and

$$\sum_{k=1}^{[n/2]}\frac{n}{k^{1+\alpha}}\int_{\pi/n}^{2\pi/n}|f(x+t+2k\pi/n)-f(x+t+(2k+1)\pi/n|\,dt=o(1)$$

uniformly in x, then the Fourier series of f is summable uniformly (C, α) $(-1<\alpha<1)$. This theorem can be used to prove a theorem on summability of Hardy and Littlewood. The author proves also: Let $f\in L$ and

$$\phi_x(t)=f(x+t)+f(x-t)-2f(x).$$

Denoting by $\sigma_n^\delta(x)$ the nth Cesàro mean of order δ of the Fourier series of f, one has

$$\sum_{n=1}^{\infty} n^{-r}|n^\delta(\sigma_n^\delta(x)-f(x))|^p\leqq(A_r p)^p\int_0^\pi|\varphi_x(t)/t^\delta|^p t^{r-2}dt$$

when $p>1$, $r>1$, $0\leqq\delta\leqq 1$, and A_r is a constant depending on r only. *R. Salem* (Cambridge, Mass.).

Izumi, Shin-ichi. Some trigonometrical series. X. Tôhoku Math. J. (2) **6**, 69–72 (1954).

Let $\varphi(t)$ be an even function of period 2π. The conditions $\int_0^t|\varphi(u)|du=o(t)$ and

$$n\int_0^{\pi/n}\left|\sum_{k=1}^{(n-1)/2}\int_{t+2k\pi/n}^{t+(2k+1)\pi/n}\{\varphi(v)-\varphi(v-\pi/n)\}v^{-1}dv\right|dt=o(1)$$

are sufficient for convergence of the Fourier series of $\varphi(t)$ at $t=0$. This extends Lebesque convergence criterion and is genuinely a stronger test as is indicated by an example. *P. Civin* (Eugene, Ore.).

Shukla, U. A theorem on the non-summability of the conjugate series of a Fourier series. Ganita **4**, 95–98 (1953).

The author replaces o by O throughout the proof of Theorem 76, pp. 66–67, of Hardy and Rogosinski's "Fourier series" [Cambridge, 1944; these Rev. **5**, 261; see also Zygmund, Trigonometrical series, Warszawa-Lwów, 1935, p. 54]. A corollary is that if

$$(*)\qquad \psi(t)=f(x+t)-f(x-t)=O(1)\quad (C, 1)\quad \text{as}\quad t\to+0,$$

then the Abel mean of the conjugate series of the Fourier series of $f(x)$ diverges to $+\infty$ if and only if

$$(**)\qquad \int_t^\pi \psi(u)\cot\tfrac{1}{2}u\,du\to+\infty\quad\text{as}\quad t\to+0.$$

It is known that (**) is sufficient even without (*) [Prasad, Ann. of Math. (2) **33**, 771–772 (1932); a related theorem of Moursund [Duke Math. J. **12**, 515–518 (1945)] was incorrectly quoted in these Rev. **7**, 60]. *L. S. Bosanquet* (London).

Abramov, L. M. On the asymptotic behavior of the Lebesgue functions of certain methods of summation of Čebyšev functions. Doklady Akad. Nauk SSSR (N.S.) **98**, 173–176 (1954). (Russian)

Let $f(x)$ be an arbitrary continuous function in the interval $-1 \leqq x \leqq +1$ and let

$$S_n(f;x)=\sum_{k=0}^{n} c_k T_k(x)$$

be the nth partial of the Fourier series of f with respect to the Čebyšev polynomials

$$T_0(x)=\left(\frac{1}{\pi}\right)^{1/2},$$

$$T_k(x)=\left(\frac{2}{\pi}\right)^{1/2}\cos k \arccos x \quad (k=1, 2, \cdots).$$

Hence $c_k=\int_{-1}^{+1} f(t)T_k(t)(1-t^2)^{-1/2}dt$ for $k>0$. Consider the "delayed arithmetic means"

$$\sigma_{n,p}(f,x)=(p+1)^{-1}\sum_{\nu=n-p}^{n} S_\nu(f,x),$$

and the corresponding "Lebesgue functions"

$$L_{n,p}(x)=\sup_{|f|\leqq 1}|\sigma_{n,p}(f;x)|.$$

Generalizing earlier results of Timan [same Doklady (N.S.) **61**, 989 (1948); Izvestiya Akad. Nauk SSSR. Ser. Mat. **14**, 85–94 (1948); these Rev. **10**, 187; **11**, 429], the author shows that for all $x \in [-1, +1]$ and $n\rightarrow\infty$ we have

$$L_{n,p}(x)=4\pi^{-2}\left\{\log\frac{1+ny}{1+(p+1)y} + |\cos ny|\log\frac{y+(n+1)^{-1}}{y+n^{-1}}\right\}+O(1),$$

where $y=\arccos x$ and the "O" is uniform in x and p $(0\leqq p\leqq n)$. *A. Zygmund* (Chicago, Ill.).

Civin, Paul. Orthonormal cyclic groups. Pacific J. Math. **4**, 481–482 (1954).

"In an earlier paper [same J. **2**, 291–295 (1952); these Rev. **14**, 163] a characterization was given of the Walsh functions in terms of their group structure and orthogonality. The object of the present note is to present a similar result concerning the complex exponentials." (Author's introduction.) *N. J. Fine* (Philadelphia, Pa.).

Merli, Luigi. Sul problema della approssimazione delle funzioni continue di due variabili. Rivista Mat. Univ. Parma **4**, 313–317 (1953).

Soient: $\xi_{\nu,n}$, $\eta_{\mu,n}$ $(\xi_{\nu,n}<\xi_{\nu+1,n};\ <\eta_{\mu,n}\eta_{\mu+1,n};\ 0\leqq n\leqq\nu,\mu;$ $n=1, 2, \cdots)$ deux suites de valeurs des variables x et y respectivement, denses dans: $0\leqq x\leqq 1$ et $0\leqq y\leqq 1$ respectivement $(0\leqq\xi_{\nu,n}, \eta_{\mu,n}\leqq 1)$; soit ensuite $\psi_{\nu,\mu}(x,y)$ un système de fonctions continues non négatives dans: $0\leqq x$, $y\leqq 1$. Pour une fonction $f(x,y)$ continue dans ce même carré on pose:

$$A_n(f)=\sum_{\mu,\nu=0}^{n} f(\xi_{\nu,n},\eta_{\mu,n})\psi_{\nu,\mu}(x,y).$$

Alors, pour que $A_n(f)$ converge vers f, quelle que soit f, il est nécessaire et suffisant que:

$$A_n(1)\rightarrow 1,\quad A_n(x+y)\rightarrow(x+y),\quad A_n[(x+y)^2]\rightarrow(x+y)^2$$

[généralisation d'un théorème de Bohman, Ark. Mat. **2**, 43–56 (1952); ces Rev. **14**, 254]. *J. Favard* (Paris).

Kaluza, Theodor, Jr. Zur Existenz stetiger grenzperiodischer Funktionen mit formal vorgegebener Fourierreihe. Arch. Math. **5**, 344–346 (1954).

As shown by Bochner [Math. Ann. **96**, 119–147 (1926)], if $f(x)\sim\sum a_\lambda e^{i\lambda x}$ is an almost periodic function and M a given module, the series $\sum_{\lambda\in M} a_\lambda e^{i\lambda x}$ is the Fourier series of an almost periodic function $f_M(x)$. The author proves this theorem, and expresses $f_M(x)$ in terms of $f(x)$ in the case when M consists of the rational multiples of a given number. *B. Jessen* (Copenhagen).

Hörmander, Lars. A new proof and a generalization of an inequality of Bohr. Math. Scand. **2**, 33–45 (1954).

Démonstration du théorème suivant, généralisation de résultats connus: Soit $f(x)$ une fonction dont le spectre (défini par la transformée de Fourier prise dans le sens de L. Schwartz) est en dehors de l'intervalle ouvert $(-\Lambda,\Lambda)$, admettant des dérivées continues jusqu'à l'ordre $(n-1)$, $f^{(n-1)}(x)$ étant absolument continue avec, presque-partout: $-M_1\leqq f^{(n)}(x)\leqq M_2$, où M_1 et M_2 sont deux nombres positifs; alors on a:

$$-\mu_1^{(n)}(M_1,M_2)\leqq\Lambda^n f(x)\leqq\mu_2^{(n)}(M_1,M_2),$$

où $-\mu_1^{(n)}$ et $\mu_2^{(n)}$ sont respectivement le minimum et le maximum dans $0\leqq x\leqq 1$, de

$$\frac{M_1+M_2}{(n+1)!}\left\{\bar{B}_{n+1}\left[x+\frac{M_2}{2(M_1+M_2)}\right]-\bar{B}_{n+1}\left[x-\frac{M_2}{2(M_1+M_2)}\right]\right\}$$

et où $\bar{B}_{n+1}(x)$ est la fonction périodique de période 1 égale au polynome de Bernoulli d'ordre $(n+1)$, $B_{n+1}(x)$, pour $0\leqq x\leqq 1$. Le résultat est étendu au cas où $M_1=\infty$, avec des conditions de croissance sur $|f|$; enfin les inégalités de Kolmogoroff sont généralisées. La démonstration, élémentaire pour le cas périodique, fait ensuite appel à la théorie des distributions de L. Schwartz. *J. Favard* (Paris).

Edwards, R. E. The exchange formula for distributions and spans of translates. Proc. Amer. Math. Soc. **4**, 888–894 (1953).

The author first establishes a number of propositions ensuring the validity of the exchange formula: $\mathfrak{F}(u*v)=U\cdot V$, where u, v are temperate distributions on R^m, the real m-dimensional Euclidean space; $U=\mathfrak{F}(u)$ and $V=\mathfrak{F}(v)$ are the Fourier transforms, and $u*v$ is the convolution distribution. The cases of interest for applications are when u, v belong to complementary L^p (R^m) spaces, or when u is a measure and v belongs to the space of continuous functions on R^m vanishing at infinity. It is for these different spaces that the author then proves a number of results on the span of translates of a given function. As an example we cite the following: There exists a positive integer k_m $(k_1=1)$ such that if $f\in L^1(R^m)$ and if E_f denotes the set of zeros of $F=\mathfrak{F}(f)$, then the space of translates of f in $L^1(R^m)$ contains every function $g\in L^1(R^m)$ whose Fourier transform G satisfies: (i) G is a function in $(\mathcal{E}^{k_m})$ (the space of functions having continuous derivatives of order $\leqq k_m$). (ii) $G=0$ on E_f and $D^qG=0$ on the derived set E_f' for every symbol of derivation of order $\leqq k_m$. For $m=1$ (ii) may be replaced by $G=0$ on E_f. It is noted by the author that for $m=1$ stronger results were obtained by Beurling [Nionde Skandinaviska Matematikerkongressen, 1938, Mercator, Helsingfors, 1939, pp. 345–366] and by Mandelbrojt and the reviewer [Acta Sci. Math. Szeged **12**, Pars B, 167–176 (1950); these Rev. **11**, 660]. *S. Agmon* (Jerusalem).

Widder, D. V. The convolution transform. Bull. Amer. Math. Soc. **60**, 444–456 (1954).

This is an exposition of results obtained in collaboration with I. I. Hirschman [Tran. Amer. Math. Soc. **66**, 135–201 (1949); **67**, 69–97 (1949); these Rev. **11**, 350]. The convolution transform of $\phi(x)$ with a kernel $G(x)$ is defined by:

$$(*) \qquad f(x)=\int_{-\infty}^{\infty} G(x-y)\phi(y)dy.$$

The inversion formula for the above is discussed when $G(x)$ is a Pólya frequency function in the terminology of I. J. Schoenberg; that is, the reciprocal of the bilateral Laplace transform of $G(x)$ is an entire function $E(x)$ belonging to the Laguerre-Pólya class and normalized so that $E(0)=1$. The main result is that giving a suitable sense to $E(D)$, where D is the differentiation operator, then the inversion of (*) is obtained by applying $E(D)$ to $f(x)$. This is illustrated in a number of special cases. *S. Agmon* (Jerusalem).

Polynomials, Polynomial Approximations

Wakulicz, A. Sur les polynômes en x ne prenant que des valeurs entières pour x entiers. Bull. Acad. Polon. Sci. Cl. III. **2**, 109–111 (1954).

The paper contains a proof of the theorem on polynomials that assume integral values for integral values of the variable [see, e.g., Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. 2, Springer, Berlin, 1925, p. 132, no. 85]. *L. Carlitz* (Durham, N. C.).

Szegö, G., and Zygmund, A. On certain mean values of polynomials. J. Analyse Math. **3**, 225–244 (1954).

Suppose $f(z)$ is a polynomial of degree n. Define for $p>0$ the quantities:

$$\|f\|_{p,a}=\left\{(2\pi)^{-1}\int_{|z|=1}|f(z)|^p|1-z^2|^a|dz|\right\}^{1/p},\quad a>0;$$

$$\|f\|_{p,L}=\left\{L^{-1}\int_{z\in L}|f(z)|^p|dz|\right\}^{1/p},$$

where L is a convex curve of length L made up of a finite number of analytic arcs meeting with maximum exterior angle $a\pi$, $1\leqq a<2$. For $q\geqq p$, let $r=1/p-1/q$. The authors establish the following inequalities:

$$\|f\|_{q,a}<A_{p,q,a}n^{(a+1)r}\|f\|_{p,a};\quad \|f\|_{q,L}<A_{p,q,L}n^{ar}\|f\|_{p,L};$$

$$\|f'\|_{p,L}<A_{p,L}\varphi(n)\|f\|_{p,L},$$

where $\varphi(n)=n^a$ if $p(a-1)>1$ and $\varphi(n)=n^a(\log n)^{a-1}$ if $p(a-1)=1$. The case where L is a finite segment corresponds to $a=2$ and that for the square to $a=3/2$. These are generalizations of well known results due to the authors and others. The proofs are from harmonic function theory, and in some cases alternate real arguments are given. Inequalities for the L_q norms of the partial sums and Cesàro means of the Fourier series of a continuous function f in terms of $\|f\|_p$ $(q\geqq p\geqq 1)$ are obtained with these methods, along with analogous results for the Laplace series of three-dimensional harmonic functions regular outside the unit sphere.

G. Klein (South Hadley, Mass.).

De Giorgi, Ennio. Un teorema sulle serie di polinomi omogenei. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **87**, 185–192 (1953).

Soit: $z=(z_1,\cdots,z_r)$, $z_h=x_h+iy_h$, $P_n(z)$ un polynome homogène de degré n; alors si

$$|P_n(z)|\leqq M \quad\text{pour}\quad y_h=0 \quad (h=1,\cdots,r),\quad \sum_1^r x_h^2\leqq R^2,$$

on a

$$|P_n(z)|\leqq(n+1)M \quad\text{pour}\quad \left(\sum_1^r x_h^2\right)^{1/2}+\left(\sum_1^r y_h^2\right)^{1/2}\leqq R.$$

De la convergence uniforme dans le champ réel d'une série de polynomes homogènes dans un voisinage de l'origine, on en tire alors l'analyticité de la somme dans un voisinage de l'origine du champ complexe. *J. Favard* (Paris).

Armstrong, James W. Point systems for Lagrange interpolation. Duke Math. J. **21**, 511–516 (1954).

Dans le problème de l'interpolation dans $[-1,+1]$ d'une fonction continue par un polynome de degré n, on sait (Bernstein, Hahn) que la constante de Lebesgue est de l'ordre de $\log n$, au moins. Suivant Erdös [Ann. of Math. (2) **43**, 59–64 (1942); ces Rev. **3**, 236] l'auteur donne des exemples de suites de noeuds d'interpolation donnant une constante de Lebesque de l'ordre $\log n$ exactement; posons:

$$-1<x_n^n<\cdots<x_1^n<1 \quad (x^n_{n+1}=-1,\ x_0^n=1);$$
$$w_n(x)=(x-x_1^n)\cdots(x-x_n^n);$$
$$M_n=\max_{[-1,1]}|w_n(x)|;\quad m_n=\min_{0\leq i\leq n}\ \max_{[x_{i+1},x_i]}|w_n(x)|;$$

alors s'il existe deux constantes positives K_1 et K_2 telles que, quel que soit n, $K_1m_n\leqq M_n\leqq K_2m_n$, la suite des noeuds $\{x_i^n\}$ répond à la question. L'auteur, après une application aux fonctions de plusieurs variables, énonce ensuite un théorème de distorsion analogue à celui de S. Bernstein [Izvestiya Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) **1931**, 1025–1050]. *J. Favard* (Paris).

Special Functions

★Lebedev, N. N. Special'nye funkcii i ih priloženiya. [Special functions and their applications.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 379 pp. 12 rubles.

This work appears to be intended as a reference book or advanced text for engineers and applied scientists. It consists of a systematic development of basic properties of many important special functions together with applications to specific problems of mathematical physics in which the author assumes a basic knowledge of the theory of functions and ordinary linear differential equations on the part of the reader. In condensed form the chapter titles are: I) The gamma function; II) The error integral; III) Exponential integrals; IV) Orthogonal polynomials; V) Cylinder functions; VI) Applications of cylinder functions; VII) Spherical harmonics; VIII) Applications of spherical harmonics; IX) The hypergeometric function; X) Functions of the parabolic cylinder. Chapter IV is devoted to the Legendre, Hermite, and Laguerre polynomials. There is no general discussion of orthogonal polynomials.

Such topics as asymptotic behavior with respect to argument and order, expansion theorems, and tables are included in addition to standard material on definitions, recurrence

formulas, and integral representations. Inter-relations among the functions are given, and some definite integrals evaluated. In general, the style is deductive rather than narrative so that the form of the exposition is similar to that in Part II of Whittaker and Watson's "Modern Analysis" [Cambridge, 1927]. Properties not derived in the text are often included in exercises, groups of which are found at the ends of several chapters.

The applications, which appear in all but Chapters I, V, VII, are diverse; but individual discussions are brief. There are, for example, problems concerning temperature distributions, electrostatic fields, mechanical vibrations, and diffraction of electromagnetic waves.

For the functions discussed lists of tables are given. In each case the range of arguments for which the function under consideration is tabulated is included. The bibliography is small as the references are mainly to books. In the reviewer's opinion this is a useful and readable book. Its contents are carefully developed and are broad enough in scope to have it reach a wide audience.

N. D. Kazarinoff (Lafayette, Ind.).

Aissen, M. I. Some remarks on Stirling's formula. Amer. Math. Monthly **61**, 687–691 (1954).

González, M. O. Elliptic integrals in terms of Legendre polynomials. Proc. Glasgow Math. Assoc. **2**, 97–99 (1954).

The author explains the known connection between complete elliptic integrals and Legendre functions of degree $\pm 1/2$, obtains the known expansion [Erdélyi et al., Higher transcendental functions, vol. II, McGraw-Hill, New York, 1953, p. 182, equation (41); these Rev. **15**, 419] of the incomplete elliptic integral of the first kind in terms of Legendre polynomials, and deduces the corresponding expansions for the elliptic integral of the second kind, and for Weierstrass' normal form.

A. Erdélyi.

Kennedy, E. C. Approximation formulas for elliptic integrals. Amer. Math. Monthly **61**, 613–619 (1954).

Approximations by expanding the integrand and integrating term by term.

L. M. Milne-Thomson.

Watson, G. N. A reduction formula. Proc. Glasgow Math. Assoc. **2**, 57–61 (1954).

Bailey, W. N. Contiguous hypergeometric functions of the type ${}_3F_2(1)$. Proc. Glasgow Math. Assoc. **2**, 62–65 (1954).

In the first paper Watson uses a very ingenious method to construct a three-term recurrence relation for the integral

$$I_n=\int_0^1 x^\alpha(1-x)^\beta\frac{d^n[x^\alpha(1-x)^\beta]}{dx^n}dx.$$

He remarks that I_n is expressible as a terminating hypergeometric series ${}_3F_2$ with unit argument, and that his recurrence relation is a relation between contiguous hypergeometric series.

In the second paper Bailey re-writes Watson's relation in two different ways as a relation between contiguous hypergeometric series, and gives a hypergeometric proof of these relations without restricting n to be an integer. He points out that similar results are true for series of any order.

A. Erdélyi (Pasadena, Calif.).

Ragab, F. M. An integral involving a product of two modified Bessel functions of the second kind. Proc. Glasgow Math. Assoc. **2**, 85–88 (1954).

Evaluation of

$$\int_0^\infty e^{-t}t^{k-1}K_m(t)K_n(z/t)dt$$

and similar integrals.

A. Erdélyi (Pasadena, Calif.).

Ragab, F. M. Further integrals involving E-functions. Proc. Glasgow Math. Assoc. **2**, 77–84 (1954).

Evaluation of integrals involving products of elementary functions, Bessel functions, and MacRobert's E-function.

A. Erdélyi (Pasadena, Calif.).

MacRobert, T. M. Integrals involving a modified Bessel function of the second kind and an E-function. Proc. Glasgow Math. Assoc. **2**, 93–96 (1954).

Evaluation of the integrals

$$\int_0^\infty t^{k-1}K_n(t)E(zt)dt,\quad \int_0^\infty e^{-t}t^{k-1}K_n(t)E(zt)dt$$

in terms of the author's E-functions; $E(z)=E(p;\alpha_r:q;\beta_s:z)$.

A. Erdélyi (Pasadena, Calif.).

Have

***Hobson, E. W. The theory of spherical and ellipsoidal harmonics.** Chelsea Publishing Company, New York, 1955. xi+500 pp.

Photo-offset reprint of the edition of the Cambridge University Press, 1931.

Harmonic Functions, Potential Theory

Shapiro, Victor L. Subharmonic functions of order r. Proc. Amer. Math. Soc. **5**, 539–546 (1954).

Il s'agit de fonctions de classe $C^{2(r-1)}$ dans un domaine plan G, dont le Laplacien itéré $r-1$ fois est sousharmonique dans G. Résultat central: pour qu'une $F \in C^{2(r-1)}$ $(r\geq 1)$ soit sousharmonique d'ordre r dans G, il suffit que son Laplacien généralisé d'ordre r soit ≥ 0 en tout point de $G-E$ (E = fermé de capacité nulle). Application à un théorème d'unicité concernant les séries trigonométriques doubles [voir à ce sujet, M. T. Cheng, Ann. of Math. (2) **52**, 403–416 (1950); ces Rev. **12**, 174; V. L. Shapiro, Duke Math. J. **20**, 359–365 (1953); ces Rev. **15**, 306].

J. Deny (Strasbourg).

Brelot, M. Majorantes harmoniques et principe du maximum. Arch. Math. **5**, 429–440 (1954).

This paper is an elaboration of two anterior notes of the author on Green spaces [C. R. Acad. Sci. Paris **235**, 598–600, 1595–1597 (1952); these Rev. **16**, 35]. The concept of an indifferent harmonic function is introduced and its properties are developed. By definition a harmonic function in a Green space is indifferent provided that for each domain $(G_P>\lambda)$ the Dirichlet problem associated with this domain and boundary values u has solution u. Here G_P is the Green's function with pole P and λ is a positive number. In addition, the notion of a minor subharmonic function is introduced. The minor subharmonic functions are shown to be the subharmonic functions which admit indifferent harmonic majorants. Generalizations of earlier results of F. Riesz on best harmonic majorants are given. The results of the paper are applied to sequences of bounded analytic functions on a

hyperbolic Riemann surface. A generalization of the Khintchine-Ostrowski theorem is given. *M. Heins.*

Titus, C. J. A projection operator on harmonic mappings. Michigan Math. J. **2**, 91–94 (1954).

A projection operator on two-dimensional harmonic mappings is defined and used to establish the quasi-interior character of such mappings. (Matrix notation is employed. In complex notation the projection operator is the formal derivative $\partial/\partial\bar{z}$.) *P. W. Berg* (Stanford, Calif.).

Bertolini, Fernando. Sul problema di Cauchy per l'equazione di Laplace in due variabili indipendenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15** (1953), 368–375 (1954).

Bertolini, Fernando. Sul problema di Cauchy per l'equazione di Laplace in due variabili indipendenti. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16**, 10–17 (1954).

D est un domaine plan simplement connexe, de frontière $\mathfrak{F}D$ assez régulière; C est un véritable sous-ensemble de $\mathfrak{F}D$, constitué par un nombre fini d'arcs fermés. On cherche f, harmonique dans D, assez régulière à la frontière, prenant sur C des valeurs données F, dont la dérivée normale prend presque partout sur C des valeurs données G; on suppose dF/ds et G dans $L^2(C)$. La fonction f, si elle existe, s'exprime à l'aide d'un système complet de fonctions harmoniques orthonormées. On discute l'existence de f moyennant la convergence de certaines séries, et on en donne diverses expressions. Extension au cas de domaines multiplement connexes, et aux solutions de $\Delta f=\varphi$, φ donnée dans D. *J. Deny* (Strasbourg).

Jacob, Caius. Sur la résolution de certains problèmes aux limites pour le plan muni de coupures rectilignes alignées. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **1** (1950), 393–417 (1951). (Romanian. Russian and French summaries)

In chapter I a solution by quadratures is indicated for the following Dirichlet problem. For $1\leqq j\leqq p$, let (a_j, b_j) define p intervals C_j on the real axis, without points in common, let Ω stand for the plane cut along the C_j's and let $f_j^+(x)$, $f_j^-(x)$ be $2p$ bounded, integrable functions with at most a finite number of singularities. The problem consists in determining a function $F(z)=U(x, y)+iV(x, y)$, regular in Ω, including the point at infinity and such that $U(x, y)$ should be uniform and take on the upper and lower edges of C_j the values $f_j^+(x)$ and $f_j^-(x)$, respectively. As an application, the following mixed Dirichlet problem is solved in chapter II: Define the p slits C_j as before; define similarly the q slits D_k without points in common among themselves, or with the slits C_j, but all along the real axis and let Ω be the plane cut along the slits C_j and D_k. Let $f_j^+(x)$, $f_j^-(x)$, $g_k^+(x)$, $g_k^-(x)$ be $2p+2q$ functions prescribed on the upper and lower edges of C_j and D_k, respectively, and satisfying the same regularity conditions as before. The problem consists in determining a function $F(z)=U(x, y)+iV(x, y)$, regular in Ω and such that $U(x, y)$ is uniform around C_j and takes on its edges the values $f_j^+(x)$ and $f_j^-(x)$, while $V(x, y)$ stays uniform around D_k and takes on its edges the values $g_k^+(x)$ and $g_k^-(x)$, respectively. The author does not seem to be aware that most of his results can be found in Muskhelishvili, Singular integral equations [Gostehizdat, Moscow-Leningrad, 1946; English translation published by Noordhoff, Groningen, 1953, Chapters 7, 12; these Rev. **8**, 586; **15**, 434]. *E. Grosswald* (Philadelphia, Pa.).

Hitotumatu, Sin. On the Neumann function of a sphere. Comment. Math. Univ. St. Paul. **3**, 1–5 (1954).

This paper contains a discussion of the characteristic properties of the Neumann function for regions of Euclidean space of $n\geqq 2$ dimensions. Explicit formulas are given for the interior and exterior regions bounded by a sphere, together with their application to the solution of the second boundary value problem of potential theory. The author notes that formulas have been given by Kellogg for the case in which $n=3$ [Foundations of potential theory, Springer, Berlin, 1929, pp. 245–247]. *F. W. Perkins.*

***Gyunter, N. M. Teoriya potenciala i ee premenenie k osnovnym zadačam matematičeskoi fiziki. [Theory of the potential and its application to the basic problems of mathematical physics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 415 pp. 13.15 rubles.

Have original

Translation into Russian of Gunther, La théorie du potentiel et ses applications aux problèmes fondamentaux de la physique mathématique [Gauthier-Villars, Paris, 1934]. There is also a short biography of Gyunter and a list of his scientific papers.

***Spencer, D. C. Dirichlet's principle on manifolds.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 127–134. Academic Press Inc., New York, 1954. $9.00.

Have

Let M be an arbitrary n-dimensional C^∞ manifold. The author introduces the generalization

$$D(\varphi)=(d\varphi, d\varphi)+(\delta\varphi, \delta\varphi)$$

of the classical Dirichlet integral. By minimizing the functional $D(\varphi)+s(\varphi-\gamma, \varphi-\gamma)$ he solves the equation

$$d\delta\varphi+\delta d\varphi+s\varphi=s\gamma.$$

The existence of such solutions is used to construct Green's function for this and the Laplace-Beltrami equation. Green's function is then used to find a harmonic p-form "with the same boundary values" as a given p-form ψ. This problem can also be solved directly by minimizing $D(\varphi)$ over all p-forms which differ from ψ only on a compact set. *H. L. Royden* (Stanford, Calif.).

Conner, P. E. The Green's and Neumann's problems for differential forms on Riemannian manifolds. Proc. Nat. Acad. Sci. U. S. A. **40**, 1151–1155 (1954).

This paper announces results to be proved in a later paper which generalizes the classical theorems of Dirichlet and Neumann on harmonic functions in a Euclidean space to forms in an open orientable Riemannian manifold M. The following orthogonal decompositions of the closure of the space H of norm-finite forms on M can be made:

$$H=[d_e\Phi]+[\delta_e\Phi]+F;$$
$$H=[d_e\Phi]+[\delta\Phi]+F_t;$$
$$H=[d\Phi]+[\delta_e\Phi]+F_n.$$

Here F consists of the forms of H satisfying $d\phi=0$, $\delta\phi=0$, F_t those forms of F whose tangential components on the (ideal) boundary of M vanish, F_n those forms of F whose normal components vanish on the boundary; $[d\Phi]$ is the closure of the space of derived forms of H, $[d_e\Phi]$ the closure of the space of the derived forms of H with compact carriers, and $[\delta\Phi]$ and $[\delta_e\Phi]$ are similarly defined. The Laplace operators corresponding to these decompositions are given, and also the associated Dirichlet and Neumann operators. The results can be applied to give properties of

closed forms with given periods on cycles in M with compact carriers, and a generalized form of Lefschetz's duality theorem

$$H^q(M; R) \approx H^{n-q}(M, bM; R)$$

is given. *W. V. D. Hodge* (Cambridge, England).

Gaffney, Matthew P. The heat equation method of Milgram and Rosenbloom for open Riemannian manifolds. Ann. of Math. (2) 60, 458–466 (1954).

The paper is concerned with the heat equation

$$\partial\alpha_t/\partial t = -\Delta\alpha_t$$

for differential forms on any non-compact suitably differentiable manifolds with a view to obtaining basic theorems for harmonic integrals on such. It does not apply the method to Volterra integral equations, as had been done on compact manifolds by Milgram and Rosenbloom [Proc. Nat. Acad. Sci. U. S. A. **37**, 180–184, 435–438 (1951); these Rev. **13**, 160, 493], but the method of spectral representation of semigroups, which had already been applied to non-compact manifolds, but for another purpose, by K. Yoshida [Proc. Japan Acad. **27**, 540–543 (1951); these Rev. **15**, 137]. The author defines the Laplacian Δ in the equation by an expression of the form $\bar{d}_e\delta' + \bar{\delta}_e d'$ in which d' and $\bar{d}_e$ are certain versions of the familiar operator d, and δ' and $\bar{\delta}_e$ are analogous versions for the operator δ; the paper is written in such a manner that the precise scope and requisitive properties of these operators are very hard to elicit and verify from the context presented. *S. Bochner* (Princeton, N. J.).

Differential Equations

Have

★Murray, Francis J., and Miller, Kenneth S. Existence theorems for ordinary differential equations. New York University Press, New York, 1954. x+154 pp. $5.00.

Let $f(x, y)$ be an n-vectorial function depending continuously on (x, y); x is a real variable and y an n-vector $y \equiv (y_1, y_2, \cdots, y_n)$; x varies in the interval I of the real axis and y in the interval J of the cartesian n-space. The existence theorems announced in the title of the book concern only the classical Cauchy problem

(1) $y' = f(x, y)$, (2) $y(x_0) = y_0$ $(x_0 \in I;\ y_0 \in J)$.

In the first chapter the existence theorem is established under the only assumption of the continuity of f. In the second chapter the case is considered when the equation (1) is obtained by solving the vectorial equation $F(x, y, y') = 0$ by means of the standard theorem on implicit functions. In the third chapter uniqueness for problem (1), (2) is proved under the usual hypothesis that $f(x, y)$ satisfies a Lipschitz condition with respect to the vector y':

(3) $|f(x, y) - f(x, y')| \leq L|y - y'|$ $(x \in I;\ y, y' \in J)$

(L not depending on x). Under the same hypothesis for $f(x, y)$ an existence proof, based on Picard's iteration procedure, is expounded in the fourth chapter, in order to derive the usual theorems on the dependence of solutions on parameters, in particular on initial points x_0 or initial values y_0 (chapter 5).

Theorems contained in these chapters are generally expounded first for the case $n=1$, and then for arbitrary n. It could be observed that in most cases the same formal proof given in the case $n=1$ can be interpreted even in the general case merely by considering y and $f(x, y)$ as n-vectors and using very elementary notions about n-vectors. This would permit a considerable shortening of the whole treatment. The last chapters of the book are devoted to linear systems. The authors state that linear systems are the only practical class of equations for which theorems "in the large" are obtainable.

This statement, in the opinion of the reviewer, is to be considered excessive, since from Picard's iteration procedure it follows that existence theorems in the large are valid under the sole assumption that $f(x, y)$ be defined for every y and a Lipschitz condition such as (3) hold for every pair y, y'. Few mathematicians would not consider the following equations in the unknown function y:

$$y' = e^{-y^2} \sin x \quad \text{or} \quad y' = x^3 \cos^2 y$$

as "practical" cases of differential equations, for which an existence theorem in the large is valid.

On the other hand, no additional proof is needed for establishing such existence theorems in the large beyond the very simple observation that, on the above hypothesis for $f(x, y)$, the Picard iterants are defined for every $x \in I$, while the proof of Picard's theorem is still valid, in order to ensure the existence in the large, if $\max|f(x, y)|$ $(x \in I;\ y \in J)$ is replaced by $\max|f(x, y_0)|$ $(x \in I)$ [see, e.g., Goursat, Cours d'Analyse, t. II, 3e éd., Gauthier-Villars, Paris, 1918, p. 378]. Thus the proof given by the authors for existence in the large in the linear case must be considered unnecessary. The extension of this theorem to the complex case stated by the authors, is not completely clear, since it is not evident what they mean when they discuss analytic functions defined on a simple rectifiable arc. The usual statement of this theorem affirms the existence of holomorphic solutions in every simply connected domain containing the initial point x_0 where the coefficients are analytic. The proof can be obtained simply and without recourse to analytic continuation.

The Green's function matrix for the Cauchy problem for linear systems is obtained by the classical method of variation of parameters.

The last sections are devoted to the study of the polydromy of solutions of linear differential systems in the complex case, when the coefficients are uniform analytic functions in a multiply connected domain of the complex plane. The main instrument for the above study is the Jordan theorem on normal matrices. *G. Fichera* (Trieste).

Parasyuk, O. S. Ergodicity of geodesic flows on certain three-dimensional manifolds of variable negative curvature. Dopovidi Akad. Nauk Ukrain. RSR 1953, 387–388 (1953). (Ukrainian. Russian summary)

The author proves that geodesic flows are ergodic on certain 3-dimensional manifolds which are claimed to be of variable negative curvature. These manifolds are formed from the half-space $z>0$ with element

$$ds^2 = f(z)(dx^2 + dy^2 + dz^2)/z^2$$

by identifying points congruent with respect to a group G of hyperbolic motions. The function $f(z)$ is assumed to be invariant under G and subject to certain other restrictions and the manifolds are assumed to have finite volume. Details are lacking in some important places. It appears, however, to the reviewer that under the conditions imposed $f(z)$ is constant and the manifolds are in fact of constant negative curvature. For such manifolds the theorem about the ergodicity of geodesic flows is already known [E. Hopf,

Ber. Verh. Sächs. Akad. Wiss. Leipzig 91, 261–304 (1939); these Rev. 1, 243]. *Y. N. Dowker* (London).

Barrett, J. H. Differential equations of non-integer order. Canadian J. Math. 6, 529–541 (1954).

Let $L(a, b)$ be the class of complex functions summable on $[a, b]$, $\alpha=\alpha_1+i\alpha_2$, $\|\alpha\|=\max(|\alpha_1|, |\alpha_2|)$, $f(t)$ defined a.e. on $[a, b]$. The Holmgren-Riesz transform of index α is $I(\alpha; a, b|f)$; this is given by $\int_a^b f(t)\Gamma^{-1}(\alpha)(b-t)^{\alpha-1}dt$ for $0<\mathrm{Re}\,\alpha=\alpha_1$, if the integrand is summable. For $\mathrm{Re}\,\alpha\leqq 0$ and n the least positive integer $>-\mathrm{Re}\,\alpha$, one lets

$$I(\alpha; a, b|f)=D_x^n I(n+\alpha; a, x|f)$$

at $x=b$, if $I(n+\alpha; a, x|f)$ is derivable up to order $n-1$ for $|b-x|<h$ and if the nth derivative exists for $x=b$. If $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\beta\geqq 0$, $f \,\varepsilon\, L(a, b)$, then $I(\alpha; a, x|f)$ exists on $[a, b]$ (for $\mathrm{Re}\,\alpha\geqq 1$), exists a.e. on $[a, b]$ (for $\mathrm{Re}\,\alpha<1$), $I(\beta+1; a, b|I(\alpha; a, t|f))=I(\alpha+\beta+1; a, b|f)$. If $\mathrm{Re}\,\alpha>0$, if $f_n\to f$ a.e. on $[a, b]$, where $f_n, f \,\varepsilon\, L(a, b)$, and if

$$\|f_n(x)\|\leqq g(x) \,\varepsilon\, L(a, b),$$

then $I(\alpha; a, x|f_n)\to I(\alpha; a, x|f)$ a.e. on $[a, b]$ [this connects with E. J. McShane, Integration, Princeton, 1944, pp. 160–168; these Rev. 6, 43]. An effective study is made of the integral-difference equation $I(-\alpha; a, x|y)+\lambda y=h(x) \,\varepsilon\, L(a, b)$ ($\mathrm{Re}\,\alpha>0$, complex parameter λ) under conditions:

$$I(i-\alpha; a, a^+|y)=K_i \quad (i=1, \cdots, n;\ n-1\leqq \mathrm{Re}\,\alpha<n).$$

This is carried out with the aid of an inversion theorem for Holmgren-Riesz transforms. Finally, the entire function $E_\alpha(z)=\sum_{p=0}^\infty z^p/\Gamma(p\alpha+1)$ ($\mathrm{Re}\,\alpha>0$), which is of historic interest, is studied along the real axis. This paper connects with the work of M. Riesz [Acta Math. 81, 1–223 (1949); these Rev. 10, 713]. *W. J. Trjitzinsky* (Urbana, Ill.).

Ženhèn, O. On a method of proof of the solvability of Cauchy's problem. Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 143–146 (1954). (Russian)

The author proves the existence of a unique solution to the initial-value problem of the equation

$$y^{(n)}=f(x, y, y', \cdots, y^{(n-1)}) \quad (f \text{ Lipschitzian})$$

by transforming it into an integral equation of Volterra type for the function $y^{(n-1)}$, which is solved by iteration. The same procedure is applied to the initial-value problem of certain integro-differential equations. *M. Golomb.*

Gautschi, Walter. Über eine Klasse von linearen Systemen mit konstanten Koeffizienten. Comment. Math. Helv. 28, 186–196 (1954).

Let θ be a formal additive operator on a suitable class of complex vector-valued functions $\mathbf{u}(x)$ of the real variable x; let A be an $n\times n$ real constant matrix; let $\varphi(t, \lambda)=\sum_{\nu=0}^m p_\nu(\lambda)t^\nu$, $p_\nu(\lambda)$ being polynomials. The author shows that formal solutions of the system of equations

$$\varphi(\theta, A)\mathbf{u}=0 \tag{1}$$

are given by

$$\mathbf{u}(x)=\sum_{i=0}^{p-1} v_{\lambda, i}(x)(A-\lambda I)^i\mathbf{c}. \tag{2}$$

Here λ is a p-fold characteristic root of A, $\mathbf{c}$ is any solution of $(A-\lambda I)^p\mathbf{c}=0$, and the functions $v_{\lambda, i}(x)$ are solutions (assuming such exist) of

$$\sum_{j=0}^{i}\frac{1}{j!}\left[\frac{d^j}{da^j}\varphi(\theta, a)\right]_{a=\lambda} v_{\lambda, i-j}=0 \quad (i=0, 1, \cdots, p-1).$$

This generalizes results of Frame [Amer. Math. Monthly 47, 35–37 (1940)], Lindfield [ibid. 47, 552–554 (1940)] and Kumorovitz [Ann. Soc. Polon. Math. 23, 190–200 (1950); these Rev. 12, 827] where $\theta=D$, i.e., differentiation with respect to x, or $\theta=E=\Delta_h+I$, i.e., Δ is the difference operator and E the corresponding translation operator by h, and where $\varphi(t, \lambda)=t-\lambda$.

The author shows that, in case $\theta=D$ or E, if $\varphi(a, t)=0$ has simple roots $t_a\neq 0$, then the $v_{\lambda, i}$ are as expected, i.e.,

$$v_{\lambda, i}=\left[\frac{d^i}{da^i}(\exp t_a x)\right]_{a=\lambda} \quad \text{when } \theta=D$$

and

$$v_{\lambda, i}=\left[\frac{d^i}{da^i}(\exp(\log t_a x))\right]_{a=\lambda} \quad \text{for } \theta=E.$$

The former holds, of course, even if $t_a=0$. It is then shown that, in these cases, the general solution of (1) is given by the usual linear combination of the particular solutions (2) summed over the different characteristic values of A. *C. R. De Prima* (Pasadena, Calif.).

Evans, Robert L. Erratum to: Asymptotic and convergent factorial series in the solution of linear ordinary differential equations. Proc. Amer. Math. Soc. 5, 1000 (1954).

See same vol. 89–92 (1954); these Rev. 15, 527.

Almeida Costa, A. Constant and periodic matrices. Univ. Lisboa. Revista Fac. Ci. A. Ci. Mat. (2) 3, 61–74 (1954). (Portuguese)

This paper contains a few complementary remarks to a paper by F. Veiga de Oliveira [same Revista 2, 201–288 (1952); 3, 5–59 (1954); these Rev. 16, 132]. *J. L. Massera* (Montevideo).

Amato, Vincenzo. Sull'integrazione di un sistema di equazioni differenziali lineari omogenee a matrice circolante w. Matematiche, Catania 8, no. 1, 23–26 (1953).

Let $a_i(x)$, $i=1, \cdots, n$, be integrable complex-valued functions on the real x-axis; let w be a fixed complex number; then the matrix $A=\sum_{i=1}^n a_i W^{i-1}$ is said to be "w-cyclic" if $W=(w_{ik})$ is such that, for $i=1, \cdots, n-1$, $k=1, \cdots, n$, $w_{ik}=\delta_{i+1, k}$ and $w_{nk}=w\delta_{1k}$. When $w=1$ this reduces to the usual cyclic matrix. The author considers the differential equation system $dy/dx=Ay$ for $\mathbf{y}=(y_1, \cdots, y_n)$ and shows that if A is w-cyclic it can be integrated by quadratures. *C. R. De Prima* (Pasadena, Calif.).

Yoshizawa, Tarô. Note on the boundedness of solutions of a system of differential equations. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 28, 293–298 (1954).

L'A. estende i risultati di una sua precedente ricerca sulle equazioni differenziali ordinarie del second'ordine [le stesse Mem. 28, 27–32 (1953); questi Rev. 15, 624]; egli indica condizioni necessarie e sufficienti affinché per le soluzioni di un sistema differenziale ordinario del prim'ordine

$$y_i'=f_i(x, y_1, \cdots, y_n),$$

con le f_i continue per x positiva o nulla e le y qualunque, da una disuguaglianza del tipo

$$y_1^2(0)+\cdots+y_n^2(0)\leqq \alpha_n^2$$

se ne possa dedurre una del tipo

$$y_1^2(x)+\cdots+y_n^2(x)\leqq \beta_n^2 \quad (0\leqq x<+\infty),$$

con il numero β convenientemente scelto in funzione del numero α. *G. Scorza Dragoni* (Padova).

Agliata, Salvatore. Integrazione di particolari sistemi di due equazioni differenziali lineari e periodicità dei loro integrali generali. Matematiche, Catania 8, no. 1, 14–22 (1953).

The author gives a somewhat lengthy proof of the simple theorem that the complex differential equation $\dot{z}=f(t)z$, where $f(t)$ is integrable on $0\leqq t\leqq T$ and periodic with period T, has solutions of period kT when $k\geqq 1$ is an integer if and only if

$$\frac{1}{2\pi i}\int_0^{kT} f(t)dt=0 \pmod 1.$$

C. R. De Prima (Pasadena, Calif.).

Gubar', N. A. A characteristic of composite singular points of a system of two differential equations by means of simple singular points of neighboring systems. Doklady Akad. Nauk SSSR (N.S.) 95, 435–438 (1954). (Russian)

This paper deals with a real system

$$(1)\qquad \dot{x}=P(x,y),\quad \dot{y}=Q(x,y)$$

where P, Q are analytic in a certain domain G and without common factor in G. Complicated singular points in G are studied and one supposes that at any such point

$$(2)\qquad |P_x'|+|P_y'|+|Q_x'|+|Q_y'|\neq 0.$$

[Reviewer's remark: (2) restricts the singular points to types such that in the expansions of P, Q about any such point some terms of the first degree are present.] Let there be associated with (1) a system

$$(3)\qquad \dot{x}=P+p(x,y),\quad \dot{y}=Q+q(x,y)$$

with p, q analytic in G. We say that p, q are ϵ-increments of rank r whenever they and their partials of order $\leqq r$ remain $<\epsilon$ in absolute value in G. The point M of G is said to be of multiplicity m for (1) whenever there exist positive ϵ_0, δ_0 such that: (a) for all ϵ_0-increments of rank m the system (3) has no more than m singular points in the δ_0-neighborhood of M; (b) for any $\epsilon<\epsilon_0$ and $\delta<\delta_0$, there exist ϵ-increments of rank m such that correspondingly in the δ-neighborhood of M, (3) has m ordinary singular points (node, focus, saddle point).

Using these definitions the author states a number of rather complicated theorems which describe the various possible phase-portraits about M. One half of these theorems corresponding to a single characteristic root at M are consequences of the work of Bendixson [Acta Math. 24, 1–30 (1901)]. The other half (both characteristic roots zero) are new. *S. Lefschetz* (Princeton, N. J.).

Basov, V. P. Construction of solutions of a class of systems of linear differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 313–328 (1954). (Russian)

The equation dealt with is $\dot{x}=(P+t^{-\gamma}Q(t))x$, where x is an n-vector, P a constant matrix, Q a continuous and bounded matrix for $t\geqq t^*$ and γ is a positive constant. The author gives the form of a solution corresponding to any given real characteristic root ρ of P such that no other characteristic root of P has ρ for real part. The paper follows the general ideas of N. P. Erugin [Trudy Mat. Inst. Steklov. 13 (1946); these Rev. 9, 509]. *S. Lefschetz.*

Nemyckiĭ, V. V. Lyapunov's method of rotating functions for finding oscillatory regimes. Doklady Akad. Nauk SSSR (N.S.) 97, 33–36 (1954). (Russian)

A set G in euclidean n-space is said to be torus-like if it can be obtained from a topological torus T by suppressing a finite number of open subsets of T. To each point of G there may be assigned the value of an angular parameter which determines the meridian through P. Let $\dot{x}=f(x)$ (x, f are n-vectors) have a trajectory $\varphi(P,t)$ wholly immersed in G. Then φ is said to define an oscillatory régime relative to G (=O. r. G) whenever: (a) φ is stable à la Poisson; (b) there exists a $\tau>0$ such that for $0<t\leqq\tau$ the point $\varphi(P,t)$ crosses each meridian exactly once.

Let $V(x)=F_1(x)/F_2(x)$. Then V is a rotating function of Lyapunov relative to $\bar{G}$ whenever: (a) the F_i are continuously differentiable in $\bar{G}$; (b) $\bar{G}$ contains no fixed point of the pencil $F_1=CF_2$; (c) $F_2=0$ represents both $V=+\infty$ and $V=-\infty$; (d) every hypersurface of the pencil meets $\bar{G}$; (e) every point of $\bar{G}$ is on some hypersurface of the pencil. We may write $\dot{V}=\Gamma(x)\cdot C^2$, $(\dot{V})_{C=0}=B(x)$ ($\dot{V}$=time derivative along a trajectory). Theorem. Suppose that: (a) $\bar{G}$ contains no singular point; (b) the trajectories through all the boundary points of G enter G; (c) there exists V as above such that on G: $|\Gamma|>a^2\geqq 0$, $|B|>\alpha^2>0$. Then $\bar{G}$ contains an O. r. G.

The set S is called a local section of a flow if there exists a θ such that every arc $f(P,-\theta,+\theta)$ of a trajectory $\varphi(P,t)$ leaving S at $t=0$ has just one point in common with S. Theorem. Let G possess a local section S of dimension $n-1$. If every trajectory $\varphi(P,t)$ starting from S at time $t=0$ returns to it at a certain fixed time $t=\omega$, then G contains a periodic solution. [Additional references: L. L. Rauch, Contributions to the theory of non-linear oscillations, Princeton, 1950, pp. 39–88; these Rev. 11, 665; Andronov, Bautin, and Gorelik, Avtomatika i Telemehanika 7, 15–41 (1946); F. B. Fuller, Ann. of Math. (2) 56, 438–439 (1952); these Rev. 14, 556.] *S. Lefschetz* (Princeton, N. J.).

Vinograd, R. È. On an assertion of K. P. Persidskiĭ. Uspehi Matem. Nauk (N.S.) 9, no. 2(60), 125–128 (1954). (Russian)

The following assertion was made by Persidskiĭ [Izvestiya Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1 (1947)]. Given an n-vector system

$$(1)\qquad \dot{x}=A(t)x,\quad 0\leqq t\leqq\tau,$$

let $A_\tau(t)$ be the matrix $\equiv A(t)$ for $t\leqq\tau$ and continued periodically beyond τ. Let λ_i, $\lambda_i(\tau)$, $i=1,2,\cdots,n$ be the characteristic Lyapunov numbers of (1) and of

$$(2)\qquad \dot{x}=A_\tau(t)x.$$

Then a necessary and sufficient condition for the regularity of (1) [in the sense of Lyapunov, Problème général de la stabilité du mouvement, Princeton, 1947, p. 237; these Rev. 9, 34] is that $\lim\lambda_j(\tau)=\lambda_j^*$ exist as $\tau\to+\infty$, and then with proper ordering $\lambda_j^*=\lambda_j$. When $A(t)$ is triangular this is obvious, but for general $A(t)$ the author shows by an example that the assertion need not hold. [Reference: Malkin, Theory of stability of movement, Gostehizdat, Moscow-Leningrad, 1952; these Rev. 15, 873.]

S. Lefschetz (Princeton, N. J.).

Spasskiĭ, R. A. On a class of regulated systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 329–344 (1954). (Russian)

The control system under consideration leads to a set of equations

$$(1)\qquad x_i=\sum_{j=1}^{N} b_{ij}x_j+\sum_{s=1}^{m} h_{is}f_s(\sigma_s),\quad \sigma_s=\sum_{j=1}^{N} p_{sj}x_j,$$

where m is the number of regulating organs, the $f(\sigma)$ are

non-linear functions, the b, p, h are constants such that $\|b_{ij}\|$ has only simple invariant factors of which k are zero $(n+k=N)$. It is assumed that: (a) the $f(\sigma)$ are continuous and are such that whatever the initial conditions the system has a unique solution;

(b) $$f_s(0)=0, \quad \sigma_s f_s(\sigma_s)>0 \quad \text{for} \quad \sigma_s\neq 0.$$

Let the system be reduced by linear transformation to

$$\dot z_i=\lambda_i z_i+\sum g_{is}f_s(\sigma_s), \quad i=1, 2, \cdots, n,$$

(2) $$\dot z_{n+j}=\sum g_{n+j,s}f_s(\sigma_s), \quad j=1, 2, \cdots, k,$$
$$\sigma_s=\sum \gamma_{s,j}z_j.$$

Two questions are considered. (I) Necessary and sufficient conditions (n.a.s.c.) for the origin to be the sole position of equilibrium. It is proved that a necessary condition is $m\geqq k$. Furthermore, if $m=k$, then a n.a.s.c. for uniqueness of the origin in the above sense is $|\gamma_{s,n+r}|\neq 0$, $|g_{n+r,s}|\neq 0$; $s, r=1, 2, \cdots, m$. (II) The asymptotic stability of the system. This is established (using the preceding results) by constructing effectively a suitable Lyapunov function. A number of special cases (small values of m, k) are discussed explicitly. *S. Lefschetz* (Princeton, N. J.).

Duvakin, A. P., and Letov, A. M. On the stability of regulating systems with two organs of regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 163–166 (1954). (Russian)

The systems under consideration are described by the equations

(a) $$\dot x_i=\sum_{k=1}^{n}b_{ik}x_k+\sum_{j=1}^{2}u_{ij}\varphi_j\left(\sum_{k=1}^{n}g_{jk}x_k\right) \quad (i=1, \cdots, n),$$

where the b_{ik}, u_{ij}, g_{jk}, are constants, the matrix (b_{ik}) has distinct characteristic values λ_i with Re $\lambda_i<0$, the functions φ_j are continuous, $\varphi_j(0)=0$ and $y\varphi_j(y)>0$ for $y\neq 0$. The authors construct, by Malkin's method [Theory of stability of movement, Gostehizdat, Moscow-Leningrad, 1952; these Rev. 15, 873] a Lyapunov function for system (a) from which they derive a sufficient condition for the asymptotic stability of its solutions. Also considered is the case where two of the λ_i are 0. *M. Golomb* (Lafayette, Ind.).

Demidovič, B. P. On a generalization of N. N. Bogolyubov's principle of averaging. Doklady Akad. Nauk SSSR (N.S.) 96, 693–694 (1954). (Russian)

Let there be given a system

(1) $$\dot x=\epsilon X(t, x)$$

where x, X are n-vectors and $\epsilon>0$ and small. Furthermore, X satisfies the usual conditions of the Cauchy-Lipschitz theorem. Let

$$X_0(x)=\lim_{T\to+\infty}\frac{1}{T}\int_0^T X(t, x)dt.$$

Consider the system

(2) $$\dot y=\epsilon X_0(y)$$

and let $y(t)$ be the solution of (2) such that $y(0)=x_0$, where x_0 is independent of ϵ. Let $x(t, \epsilon)$ be the solution of (1) such that $x(0, \epsilon)=x_0$. It has been shown by Bogolyubov [On some statistical methods in mathematical physics, Akad. Nauk Ukrain. SSR, 1945; these Rev. 8, 37] that $x(t, \epsilon)\to y(t)$ as $\epsilon\to 0$, and this uniformly on any finite interval of length $O(1/\epsilon)$. This result is extended by the present author to a system (1) in which ϵX is replaced by a function $X(t, x, \epsilon)$. [Additional references: Los', Ukrain. Mat. Ž. 2, no. 3, 87–93 (1950); these Rev. 14, 47; Fatou, Bull. Soc. Math. France 56, 98–139 (1928)]. *S. Lefschetz.*

Demidovič, B. P. On some averaging theorems for ordinary differential equations. Mat. Sbornik N.S. 35(77), 73–92 (1954). (Russian)

This contains the proof and further elaboration of a theorem stated in the paper reviewed above. The reference is to conditions under which a solution of

$$\dot x=f(t; x; \epsilon) \qquad (x, f: n \text{ vectors})$$

tends, as $\epsilon\to 0$, to the solution with the same initial value of $\dot x=\bar f(x)$, where

$$\bar f(x)=\lim_{\epsilon\to 0}\frac{1}{T}\int_{t_0}^{t_0+T}f(t; x; \epsilon)dt, \quad T>0,$$

and is assumed independent of T. Analogous theorems are proved for systems

(4) $$\dot x=P(t, \epsilon)x+Q(t, \epsilon),$$
(5) $$\dot x=P(t, \epsilon)x.$$

Applications are also made to the case where P, Q are periodic or almost periodic. *S. Lefschetz.*

Antosiewicz, H. A., and Davis, P. Some implications of Liapunov's conditions for stability. J. Rational Mech. Anal. 3, 447–457 (1954).

The main result of the paper is: If $X(t, t_0)$ is the solution of $dX/dt=A(t)X$ for which $X=I$ when $t=t_0\geqq 0$, $A(t)$ being continuous and bounded for $t\geqq 0$, then $\|X(t, t_0)\|\leqq Me^{-\alpha(t-t_0)}$ for $t\geqq t_0$ (with M, α independent of t_0) if and only if there exists a quadratic form $V(x, t)$ in x, with continuously differentiable coefficients, such that

$$c_0\|x\|^2\leqq V(x, t)\leqq c_1\|x\|^2$$

and

$$\frac{\partial}{\partial t}V(x, t)+\text{grad } V(x, t)\cdot A(t)x\leqq -c_2\|x\|^2$$

(with $c_i>0$, $i=1, 2, 3$). *G. E. H. Reuter* (Manchester).

Kamenkov, G. V., and Lebedev, A. A. Remarks to a paper on stability over a finite interval of time. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 512 (1954). (Russian)

A modification of definitions previously given by the authors [Kamenkov, same journal 17, 529–540 (1953); these Rev. 15, 795; Lebedev, ibid. 18, 139–148 (1954); these Rev. 16, 132] is proposed: consider a cycle $V(t, x)=A$ such that the diameter $D(t)$ of $V\leqq A$ is $\leqq D(t_0)$; then the solution $x=0$ of a dynamical system is stable over $[t_0, t_0+\tau]$ if for any x_0 such that $V(t_0, x_0)\leqq A$ we have $V(t, x)\leqq A$ for $t\in[t_0, t_0+\tau]$. The new definition makes certain changes necessary in the theorems proved in the above mentioned papers. *J. L. Massera* (Montevideo).

Eršov, B. A. A theorem on stability of motion in the large. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 381–383 (1954). (Russian)

It was proved by Krasovskiĭ [same journal 18, 95–102 (1954); these Rev. 15, 873] that the origin is asymptotically stable in the large for the system

(1) $$\dot x=F(x, y), \quad \dot y=f(\sigma), \quad \sigma=cx-dy,$$

where $F(0, 0)=f(0)=0$ and c, d are constants, not both zero.

The assumed restrictions were:

$$(2)\qquad \sigma\neq 0:\quad \sigma f(\sigma)>0,\quad \sigma[\varphi(\sigma,y)-\varphi(0,y)]<0;$$

$$(3)\qquad y\neq 0:\quad y\varphi(0,y)<0;$$

$$(4)\qquad \left|\int_0^{\pm\infty} f(\sigma)\,d(\sigma)\right|=+\infty,\quad \left|\int_0^{\pm\infty} \varphi(0,y)\,dy\right|=+\infty,$$

where

$$\varphi(\sigma,y)=cF\left(\frac{\sigma+dy}{c},y\right)-df(\sigma).$$

The author shows that the same property holds without any regard for (4). *S. Lefschetz* (Princeton, N. J.).

Vasil'eva, A. B. On differentiability of solutions of systems of differential equations containing small parameters with the derivatives. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9, no. 3, 29–40 (1954). (Russian)

Let there be given a system

$$(1)\qquad \begin{aligned}\mu\dot z&=F(z,y,t)\\ \dot y&=f(z,y,t)\end{aligned}$$

where z, F are n-vectors and y, f are m-vectors, and where μ is a small parameter. Consider the associated degenerate system

$$(2')\quad F(z,y,t)=0;\qquad (2'')\quad \dot y=f(x,y,t),$$

it being assumed that (2′) is algebraic in z. Thus (2) may be replaced by a finite set of relations

$$(3)\qquad \dot y=f(\varphi(y,t),y,t).$$

It was proved in various papers [e.g., Vasil'eva, Mat. Sbornik N.S. **31**(73), 587–644 (1952); these Rev. **14**, 1086] that under certain conditions and for a certain range $(0,T)$ the solutions of (1) tend to certain solutions of (2). In the last paper cited the author calculated the derivatives $dz(t,\mu)/d\mu$, $dy(t,\mu)/d\mu$ for μ small on the basis of Lyapunov's second method. The same result is now obtained in relation to Lyapunov's first method, i.e., in relation to the characteristic roots of the Jacobian matrix $[\partial F/\partial z]_{z=\varphi(y,t)}$. The result is extended to the case of several vectors z and as many variables μ. *S. Lefschetz* (Princeton, N. J.).

Vasil'eva, A. B. On the mathematical theory of catalysis. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9, no. 6, 39–46 (1954). (Russian)

A catalytical decomposition process symbolized by the equations $Y+Z\rightarrow M+N_1$, $M\rightarrow Z+N_2$, where Y represents the substance decomposed, Z the catalyser and N_1, N_2 substances which disappear as gases or precipitates, is governed by the system $\dot y=-kyz$, $\dot z=-kyz+am$, $\dot m=kyz-am$, $y(0)=y_0$, $z(0)=z_0$, $m(0)=0$. If $\zeta=z/z_0$, $\eta=y/y_0$, $\mu=z_0/y_0$, $\lambda=a/ky_0$, the system reduces to $\mu d\zeta/d\eta=1-\lambda(1-\zeta)/\eta\zeta$, $\zeta(1)=1$, which is an equation with a small coefficient in the highest derivative. The author studies the question of the approximation of the "limiting solution" (for $\mu=0$) $\bar\zeta=\lambda/(\lambda+\eta)$ and introduces certain corrections which are important, especially in the neighborhood of the initial point $\eta=1$. The result appears to be satisfactory in two numerical cases considered. *J. L. Massera.*

Belyustina, L. N. On conditions for the existence of a center. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 511 (1954). (Russian)

The author gives necessary and sufficient conditions for a center at the origin of

$$\frac{dy}{dx}=\frac{-x+Ax^2+Bxy+Cy^2}{y+Cx^2+Dxy+Ey^2}.$$

By means of a rotation of the axes the system is reduced to the case $C=A$ already dealt with before by Kapteyn [Akad. Wetensch. Amsterdam. Verslagen, Afd. Natuurkunde **19**, 1446–1457 (1911); **20**, 1354–1365; **21**, 27–33 (1912)], Dulac [Bull. Sci. Math. (2) **32**, 230–252 (1908)], Bautin [C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 669–672 (1939); these Rev. **2**, 49], and Sakharnikov [Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 669–670 (1948); these Rev. **10**, 377]. *S. Lefschetz* (Princeton, N. J.).

✶Belyustina, L. N. On conditions for the existence of a center. Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 2 pp. (mimeographed). $1.00.

Translation of the paper reviewed above.

Müller, R. Lösung spezieller gewöhnlicher Differentialgleichungen durch unendliche Operatoren. Anwendung auf das ballistische Problem. Z. Angew. Math. Mech. 34, 273–274 (1954).

This author reduces the ballistic equations of Euler (for direct fire), namely

$$\dot v=-cf(v)-g\sin\theta,\quad v\dot\theta=-g\cos\theta,$$

to the form

$$\frac{d}{dv}(v\dot v)^2=2v(g^2-(cf)^2)\pm\sqrt{[(v\dot v)^2]}2c(2f+vf'),$$

which he proposes to solve for $(v\dot v)^2$ as a function of v by an unending sequence of quadratures. *A. A. Bennett.*

Fage, M. K. The characteristic function of a one-point boundary problem for an ordinary linear differential equation of second order. Doklady Akad. Nauk SSSR (N.S.) 96, 929–932 (1954). (Russian)

This is a continuation of earlier work by the same author [same Doklady (N.S.) **95**, 721–724; erratum **97**, 572 (1954); these Rev. **16**, 264]. The characteristic function

$$R(w;x,s)=\sum_{n=1}(3n)!A_n(x,s)/(w^{3n+1}\cdot n!),$$

$0\leqq s\leqq x$, of the problem $L(y)=y''+p_1(x)y'+p_2(x)y=f(x)$, $y(0)=y'(0)=0$, $0\leqq x\leqq X\leqq\infty$, x bounded even if $X=\infty$, A_n being the nth iterate of the corresponding Green's function, satisfies $-3w^{-1}\cdot(\partial^2R/\partial w^2)+L_x(R)=0$. It may be shown that R admits an analytic continuation in the whole w-plane, with the exception of the segments Δ_x^s: $\arg w=2\pi m/3$ $(m=0,1,2)$, $0\leqq|w|\leqq[3^{3/2}(x-s)/2]^{2/3}$. The set Δ which is the union of the Δ_x^s, $0\leqq s\leqq x\leqq X$, is the spectrum of the problem. *J. L. Massera* (Montevideo).

Ryškov, S. S. On the regimes of operation of a vacuum-tube generator. Doklady Akad. Nauk SSSR (N.S.) 96, 921–924 (1954). (Russian)

Several cases of the second-order equation

$$\ddot x+x=\mu F(x,\dot x,\mu)$$

are considered for small $|\mu|$ using the familiar formulation $d\rho/d\phi=r(\rho,\phi,\mu)$, where $x=\rho\cos\phi$, $\dot x=\rho\sin\phi$. *N. Levinson* (Cambridge, Mass.).

✶Ryshkov, S. S. On the operating regime of a tube oscillator. Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 4 pp. (mimeographed). $3.00.

Translation of the paper reviewed above.

Kac, A. M. Biharmonic oscillations of a dissipative non-linear system which are induced or sustained by a harmonic disturbing force. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 425–444 (1954). (Russian)

The author discusses approximate solutions of

$$(1)\qquad \ddot{x}+k\dot{x}+x+x^3=R\sin m\omega t,$$

where k, R, ω are constants and m is an integer. Approximate solutions of type

$$(2)\qquad x=A\sin(m\omega t+\varphi)$$

have already been treated [for instance, in the author's paper, Trudy Leningrad. Indust. Inst. no. 3 (1939)]. Here he turns his attention to more complicated approximations of type

$$(3)\qquad x=C+A\cos(m\omega t+\varphi)+B\sin(n\omega t+\psi),$$

where m, n are relatively prime integers. Writing (3) as

$$(4)\qquad \begin{gathered} x=Cf_1+a_1f_2+a_2f_3+b_1f_4+b_2f_5,\\ f_1=1,\quad f_2=\sin m\omega t,\quad f_3=\cos m\omega t,\\ f_4=\sin n\omega t,\quad f_5=\cos n\omega t, \end{gathered}$$

the coefficients are determined so that the Fourier expansion of $F(x,\dot{x},\ddot{x},t)$, the difference of the two sides of (1), contains no constant term, and no mth and nth harmonics. In substance there results the approximation (2) except in the following four cases which are discussed in detail and are characterized by these values of (m,n): (1, 3), (3, 1), (1, 2), (2, 1). [Additional references: Krylov and Bogolyubov, Introduction to non-linear mechanics, Izdat. Akad. Nauk USSR, Kiev, 1937; Lurye and Čekmarev, Akad. Nauk SSSR. Prikl. Mat. Meh. (N.S.) 1, 307–324 (1938).]

S. Lefschetz (Princeton, N. J.).

★Minorsky, N. On the stroboscopic method. Studies in mathematics and mechanics presented to Richard von Mises, pp. 192–199. Academic Press Inc., New York, 1954. $9.00.

Brief resumé of the method discussed at greater length in Bull. Soc. Franç. Méc. 4, no. 13, 15–26 (1954); these Rev. 16, 131.

A. S. Householder (Oak Ridge, Tenn.).

Slezinger, I. N. Motion of a very simple mechanical system under the action of elastic forces and nonlinear friction. Akad. Nauk SSSR. Žurnal Tehn. Fiz. 24, 1660–1676 (1954). (Russian)

The author studies certain non-linear processes with applications, for instance, to the performance of heavy-duty lathes. Consider a heavy body of mass m elastically connected to a driving body which moves under a (rather small) constant speed v and subject to a non-linear frictional resistance $R(\dot{x})$ (it is assumed that R starts, for $\dot{x}=0$, from a positive value, then decreases to a minimum at $x=\dot{x}_0$ and then increases; as a consequence, no motion takes place if the absolute value of the driving force is less than $R(0+)$). The equation of motion is then $m\ddot{x}+R(\dot{x})\operatorname{sgn}\dot{x}-P(x,t)=0$. The author replaces the graph of R by an approximation consisting of a segment with negative slope, from $\dot{x}=0$ to $\dot{x}=\dot{x}_0$, and a ray with positive slope from $\dot{x}=\dot{x}_0$ to $+\infty$. A rather laborious discussion shows that the motion may be jerky (with intervals of rest), oscillatory or steady (after a few initial damped oscillations); approximate equations are given for the determination of the values of the parameters which separate one kind of motion from another.

J. L. Massera (Montevideo).

De Castro, Antonio. Un teorema di confronto per l'equazione differenziale delle oscillazioni di rilassamento. Boll. Un. Mat. Ital. (3) 9, 280–282 (1954).

The author considers the equations

$$(\mathrm{A})\qquad \ddot{x}+f_1(x,\dot{x})\dot{x}+g(x)=0,$$

$$(\mathrm{B})\qquad \ddot{x}+f_2(x,\dot{x})\dot{x}+g(x)=0,$$

where f_1, f_2, g are continuous and satisfy Lipschitz conditions in every bounded domain, $xg(x)>0$ for $x\neq 0$, and $\int_0^{\pm\infty}g(x)dx=+\infty$. It is shown that if (A) has a periodic solution, and if $f_2(x,\dot{x})>f_1(x,\dot{x})$ and $f_2(0,0)<0$, then (B) has a periodic solution, such that if C_1 and C_2 are the closed curves representing the periodic solutions of (A) and (B), respectively, in the usual phase plane, C_2 is interior to C_1. Some simple extensions of this result are indicated.

L. A. MacColl (New York, N. Y.).

Kauderer, H. Zur Analyse der Dämpfung freier Schwingungen. Ing.-Arch. 22, 251–257 (1954).

Given a harmonic oscillator with small nonlinear damping depending on the velocity only, the problem is to find the analytic form of the damping force as a function of velocity in order to account for a given experimentally measured set of oscillation amplitudes. The solution is not unique, of course, but the coefficients in a hypothetical polynomial solution can be determined by these data, using, say, the method of least squares. The formulas connecting the coefficients and the oscillation amplitudes are derived by the method of Kryloff-Bogoliuboff.

E. Pinney.

Stampacchia, Guido. Sopra una generalizzazione dei problemi ai limiti per i sistemi di equazioni differenziali ordinarie. Ricerche Mat. 3, 76–94 (1954).

Let $i, j, k=1,\cdots,n$, let C denote the slab $a\leqq x\leqq b$, $|y_i|<\infty$, and let V_i denote $(n-1)$-dimensional manifolds in C. The problem (P) is to find in C solutions of (E_1) $y_i'=f_i(x,y_k)$ intersecting all the V_i; the f_i are to be continuous in the y_k and measurable in x. It is assumed that there exist summable $\varphi_i(x)$ and $\psi_{ij}(x)$ such that, in C, $|f_i-\sum\psi_{ij}y_j|\leqq\varphi_i$. First the solutions of

$$(\mathrm{E}_\lambda)\qquad y_i'=\lambda f_i+(1-\lambda)\sum\psi_{ij}y_j$$

are used to "project" V_k into a hyperplane $x=c$, yielding manifolds $U_{k\lambda}$. It is assumed that U_{k0} have a common point and then shown by topological methods, under suitable assumptions, that $U_{k\lambda}$ have a common point for $0\leqq\lambda\leqq 1$; the case $\lambda=1$ yields a solution for (P). Next, the problem $(\mathrm{E}_1,\mathrm{B})$, with linear boundary conditions (B) $\sum_j\gamma_{ij}y_j(x_i^0)=\beta_i$ is shown to be consistent if the linear homogeneous problem $(\mathrm{E}_0,\mathrm{B})$ has no nontrivial solution. Finally, if the manifolds V_i arise from linear manifolds (cf. B) by a sufficiently small homotopy (Fréchet distance), then P is consistent if $(\mathrm{E}_0,\mathrm{B})$ has no nontrivial solution. Applications include Nicoletti's problem and a nonlinear eigenvalue problem.

F. A. Ficken (Knoxville, Tenn.).

★Laasonen, Pentti. Über die Näherungslösungen der Sturm-Liouvilleschen Eigenwertaufgabe. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 176–182 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The reduced form $y''(x)+[\lambda-q(x)]y(x)=0$, of the Sturm-Liouville differential equation is written in the form $(y'/p)'+py=q'y'/(2p^3)$, where $p(x)=[\lambda-q(x)]^{1/2}$. For large values of the parameter λ the right-hand member of that second form is neglected to obtain an equation $(z'/p)'+pz=0$

whose general solution is $z=C_1\cos r+C_2\sin r$, where $r(x)=\int_0^x p(t)dt$ and C_1 and C_2 are constants. Sturm-Liouville boundary conditions are introduced at the ends of the interval $0\leqq x\leqq 1$. The author shows that the characteristic functions and numbers that are obtained by using the function $z(x)$ above in place of $y(x)$, are approached asymptotically by the characteristic functions and numbers of the original problem in $y(x)$, as λ_n grows large. Thus a fairly simple approximation to the solutions of the Sturm-Liouville problem is obtained, good at least for sufficiently large values of the characteristic numbers. A simple example is given in which the approximation is good for $n>1$.

R. V. Churchill (Ann Arbor, Mich.).

Bauer, W. F. Modified Sturm-Liouville systems. Quart. Appl. Math. 11, 273–283 (1953).

Certain problems in heat conduction lead to the following eigenvalue problem (A) for Sturm-Liouville equations:

$$-y''+p(x)y=\lambda y\quad(0<x<1),$$
$$y'(0)=(b_1+b_2\lambda)y(0),\quad y'(1)=(c_1+c_2\lambda)y(1)$$

(all functions and constants real). It is assumed that p is bounded and that $b_2\geqq 0$, $c_2\geqq 0$. A proof is given by means of Laplace transforms that there is a complete set of eigenfunctions associated with this problem. [Reviewer's remark. A simpler proof based on general principles may be obtained as follows. Consider the quadratic forms

$$\alpha(f,f)=\int_0^1(f'^2+pf^2)dx-b_1f(0)^2-c_1f(1)^2$$

and

$$\beta(f,f)=\int_0^1 f^2dx+b_2f(0)^2+c_2f(1)^2.$$

Then for large enough t, the form $\alpha_t(f,f)=\alpha(f,f)+t\beta(f,f)$ is bounded from above and from below by a constant times $|f|^2=\int_0^1(f'^2+f^2)dx$ and may serve as a norm square in the Hilbert space H obtained by closing $C^1(0,1)$ with respect to $|f|$. In this space $\beta(f,f)=\alpha_t(G_tf,f)$ $(f,G_tf\in H)$ defines a completely continuous positive self-adjoint operator G_t. It has a complete set of eigenfunctions φ_k $(k=1,2,\cdots)$ with eigenvalues $(\lambda_k+t)^{-1}$ which satisfy the boundary conditions and can be orthonormalized with respect to β. These eigenfunctions are also eigenfunctions of (A), the eigenvalue of φ_k being λ_k [cf. Gårding, Math. Scand. 1, 55–72 (1953); these Rev. 16, 366].]

L. Gårding (Lund).

Partial Differential Equations

***Tihonov, A. N., i Samarskiĭ, A. A. Uravneniya matematičeskoĭ fiziki. [The equations of mathematical physics.] 2d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 679 pp. 14.30 rubles.

For a review of the first edition [1951] see these Rev. 15, 430. Changes in this edition are minor (but include a new appendix to ch. VI on the method of finite differences in the heat-conduction equation).

Have

***Miller, Kenneth S. Partial differential equations in engineering problems.** Prentice-Hall, Inc., New York, 1953. viii+254 pp. $7.35.

This is a textbook on the solution of boundary-value problems in partial differential equations by the method of separation of variables and expansion of prescribed functions in terms of characteristic functions. Assuming that the reader has an elementary knowledge of ordinary differential equations, the author includes brief explanations of Fourier series and integrals and some of their generalizations and of other relevant topics. The chapter headings read: Derivation of partial differential equations; Fourier series; Separation of variables; The Fourier integral; Legendre, Bessel, and Mathieu functions; Properties of second-order partial differential equations; References. The partial differential equations that are introduced include those on vibrations of strings, bars, beams and membranes, the heat equation, the telegraph equation, Laplace's equation, and a few equations from hydrodynamics and elasticity. The illustrative problems are based on the simpler physical problems. Each chapter ends with a list of exercises for the reader. Answers are not given to many of those exercises. The author aims at a formal treatment; expansion theorems are stated without proof. Sturm-Liouville systems are mentioned near the end of the book; thus they are not available for clarifying the process of separation of variables. On the whole, the exposition is good. The features of including Mathieu functions and their applications, as well as the large variety of important partial differential equations, seem especially attractive.

R. V. Churchill (Ann Arbor, Mich.).

Moisil, Gr. C. Théorie préliminaire des systèmes d'équations aux dérivées partielles linéaires aux coefficients constants. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 319–401 (1952). (Romanian. Russian and French summaries)

The author considers linear systems of partial differential equations:

$$(*)\qquad \sum_j P_{ij}\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_r}\right)\varphi_j=0,$$

where the $P_{ij}(X_1,\cdots,X_r)$ are polynomials, and thus each such system is characterized by the matrix of polynomials P_{ij}. The object of this paper is to study systems by means of the properties of the associated matrix of polynomials. In particular, attention is paid to the differential relations which the solutions of a system (*) must satisfy, and also the question of the representation of solutions of (*) by means of "potentials". These considerations form the algebraical preliminaries of the first part of the paper. The second part contains a host of particular examples of systems from various branches of mathematical physics, which are treated from the above point of view. Among these are the equations for an elastic body with transverse isotropy, the equations of thermo-elastic equilibrium, the equilibrium equations for certain visco-elastic bodies, to mention only a few.

J. B. Dias (College Park, Md.).

Moisil, Gr. C. Sur les équations de la distribution spatiale instantanée des grandeurs physiques, dans le cas des phénomènes non stationnaires. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 481–496 (1952). (Romanian. Russian and French summaries)

Consider a system of linear partial differential equations with constant coefficients

$$\sum_j P_{ij}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},\frac{\partial}{\partial t}\right)\varphi_j=0,$$

where the P_{ij} are polynomials. The author is concerned with "linear differential consequences" of this system which do not involve differentiations with respect to the time, i. e.

equations of the form

$$(*)\qquad \sum_j S_j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\varphi_j=0,$$

where

$$S_j(X, Y, Z)=\sum_i U_i(X, Y, Z, T)P_{ij}(X, Y, Z, T),$$

and the $U_i(X, Y, Z, T)$ are polynomials in X, Y, Z, T. An equation like (*) is termed an "equation of condition" whereas an equation involving time differentiation is called an "equation of evolution". The determination of the equations of condition implied by a given system leads to a linear system of equations, where both the "unknowns" and the coefficients are polynomials, and in this way the author discusses various systems of partial differential equations with constant coefficients of mathematical physics. For example, the only equations of condition of Maxwell's equation in free space:

$$\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}+\mathrm{rot}\,\mathbf{E}=0,\quad \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}-\mathrm{rot}\,\mathbf{H}=0,\quad \mathrm{div}\,\mathbf{H}=0,\quad \mathrm{div}\,\mathbf{E}=0,$$

are those derivable from the last two equations.

J. B. Diaz (College Park, Md.).

Pini, Bruno. Precisazioni a un ragionamento contenuto in una mia Nota sulle equazioni a derivate parziali di tipo ellittico. Ricerche Mat. **3**, 3–12 (1954).

An earlier result [Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 176–195 (1952); these Rev. **15**, 712] is generalized and an error in the proof is rectified.

F. A. Ficken (Knoxville, Tenn.).

Magenes, Enrico. Sui problemi al contorno misti per le equazioni lineari del secondo ordine di tipo ellittico. Ann. Scuola Norm. Super. Pisa (3) **8**, 93–120 (1954).

Soit E l'opérateur différentiel

$$\sum_{h,k} a_{hk}\partial^2/\partial x_h\partial x_k+\sum_h b_h\partial/\partial x_h+c,$$

a_{hk}, b_h, c étant des fonctions suffisamment régulières dans un ouvert A du plan; on suppose l'opérateur E elliptique (et dét $(a_{hk})=1$). On donne un domaine régulier $\mathfrak{D}\subset A$, de frontière $\mathfrak{FD}=\mathfrak{F}_1\mathfrak{D}\cup\mathfrak{F}_2\mathfrak{D}$. On cherche u solution de $E(u)=0$ dans $\mathfrak{D}-\mathfrak{FD}$, u continûment dérivable dans $\mathfrak{D}-\mathfrak{F}_1\mathfrak{D}$, $\partial u/\partial\nu=0$ sur $\mathfrak{F}_2\mathfrak{D}$ ($\partial/\partial\nu$=dérivée conormale), $u=\mu$ en moyenne quadratique sur $\mathfrak{F}_1\mathfrak{D}$ (problème aux limites de type mêlé). Sous des hypothèses (U) assez compliquées, la solution de ce problème, si elle existe, est unique.

Théorème: Sous les hypothèses (U), le problème ci-dessus admet une solution (évidemment unique). Plan de la démonstration: a) Par utilisation d'un théorème de Fichera [Ann. Mat. Pura Appl. (4) **27**, 1–28 (1948); ces Rev. **10**, 534] on construit une suite u_n de solutions de $E(u_n)=0$, telles que $u_n\to\mu$ dans $L^2(\mathfrak{F}_1\mathfrak{D})$ (espace des fonctions de carré sommables sur $\mathfrak{F}_1\mathfrak{D}$) et $\partial u_n/\partial\nu\to 0$ dans $L^2(\mathfrak{F}_2\mathfrak{D})$. b) Supposons que u_n demeure dans un ensemble borné de $L^2(\mathfrak{D})$. Alors on peut montrer que $u_n\to u$ dans $L^2(\mathfrak{D})$. Il en résulte par un théorème de Fichera [Ann. Scuola Norm. Super. Pisa (3) **1**, 75–100 (1949); ces Rev. **11**, 724] que u_n converge uniformément sur tout compact de $\mathfrak{D}-\mathfrak{FD}$, ainsi que ses dérivées du premier ordre, vers u, solution régulière de $E(u)=0$. [Note du reviewer: ou encore, u est solution au sens de la théorie des distributions de Schwartz [Théorie des distributions, t. I, II, Hermann, Paris, 1950, 1951; ces Rev. **12**, 31, 833] et donc E étant elliptique, u est nécessairement solution régulière [cf. aussi Friedrichs, Comm. Pure Appl. Math. **6**, 299–326 (1953); ces Rev. **15**, 430; et John, ibid. **6**, 327–335 (1953); ces Rev. **15**, 431].] c) La fonction u est égale à μ en moyenne sur $\mathfrak{F}_1\mathfrak{D}$. On peut ensuite démontrer que l'on a $\partial u/\partial\nu=0$ sur $\mathfrak{F}_2\mathfrak{D}$. d) Reste à démontrer que u_n demeure dans un ensemble borné de $L^2(\mathfrak{D})$. On montre qu'il en est nécessairement ainsi, en utilisant l'unicité.

J. L. Lions (Nancy).

Nitsche, Johannes. Beitrag zum Randwertproblem eines linearen elliptischen Differentialgleichungssystems im Grossen. Rend. Circ. Mat. Palermo (2) **3**, 109–114 (1954).

Let the system

$$(*)\qquad u_x-v_y=a^1u+b^1v+c^1,\quad u_y+v_x=a^2u+b^2v+c^2,$$

be defined in a region T with boundary S and suppose the boundary condition $u\cos\gamma(s)+v\sin\gamma(s)=f(s)$ is imposed. The functions $a^1, \cdots, c^2, f, \gamma$ have Hölder-continuous first derivatives. It is shown that each solution which is continuous in $T+S$ and continuously differentiable in T has Hölder-continuous second derivatives in T. It is also shown that the above problem is equivalent to the same problem for a second-order linear elliptic equation for one unknown function.

M. H. Protter (Berkeley, Calif.).

Nitsche, Johannes. Eine charakteristische Eigenschaft der Lösungen von Randwertproblemen elliptischer Differentialgleichungssysteme. Arch. Math. **6**, 18–24 (1954).

Let T be a simply connected domain in the (x, y)-plane, bounded by a curve S having continuous curvature. Let $\sigma(s)$ be a Hölder-continuous function of arc on S, with $\sigma(s+L)-\sigma(s)=2\pi n$, where L is the length of S and n is a non-negative integer. Denote by $\tau(s)$ another such function of arc, with $\tau(s+L)-\tau(s)=2\pi n$ and $\tau(s)\not\equiv\sigma(s)$. Consider the boundary-value problem defined by the relations

$$\left.\begin{aligned}&(1)\qquad u_x-v_y-au-bv=f(x, y)\\&(2)\qquad u_y+v_x-cu-dv=g(x, y)\end{aligned}\right\}\text{ in } T$$

$$(3)\qquad u\cos\sigma(s)+v\sin\sigma(s)=r(s)\quad\text{on}\quad S$$

$$(4)\qquad \oint_S\{u\cos\tau(s)+v\sin\tau(s)\}h_\alpha(s)ds=c_\alpha\qquad(\alpha=0, 1, \cdots, 2n),$$

where the $\{h_\alpha(s)\}$ denote the trigonometric functions

$$1, \cos\frac{2\pi}{L}s, \sin\frac{2\pi}{L}s, \cdots, \cos\frac{2n\pi}{L}s, \sin\frac{2n\pi}{L}s.$$

The author proves the theorems: I) The homogeneous problem ($f=g=r=0$, $c_0=c_1=\cdots=c_{2n}=0$) admits only the trivial solution $u=v=0$. II) The inhomogeneous problem admits exactly one solution pair (u, v). III) If $f=g=0$, every solution pair (u, v) of (1) and (2) in T has at most $2n$ common zeros on S.

The author observes that II) is a consequence of I) and proves I) by a contradiction, obtained by suitably transforming a supposed non-trivial solution (u, v) and applying standard existence and uniqueness theorems to the transformed equations. III) appears as a corollary of the method of proof. The author states that his results are of importance in the study of boundary-value problems for non-linear differential equations.

R. Finn (Los Angeles, Calif.).

Gårding, Lars. Dirichlet's problem for linear elliptic partial differential equations. Math. Scand. 1, 55–72 (1953).

This paper presents proofs of results on Dirichlet's problem for elliptic partial differential equations previously announced by the author [C. R. Acad. Sci. Paris **233**, 1554–1556 (1951); these Rev. **14**, 174] and established independently by M. I. Višik [Mat. Sbornik N.S. **29**(71), 615–676 (1951); these Rev. **14**, 174] and the reviewer [Proc. Nat. Acad. Sci. U. S. A. **38**, 230–235, 741–747 (1952); these Rev. **14**, 174, 473]. A basic part of the argument is Theorem 2.1 which asserts the semi-boundedness of the general linear elliptic differential operator over the domain of the infinitely differentiable functions with compact supports in a given bounded open subset of Euclidean space. In the case of constant coefficients, these operators are actually positive but the author gives an example of a fourth-order operator with variable coefficients for which positivity does not hold. Remarks are made toward the end of the paper concerning a generalized Neumann's problem and the vibration problem. *F. Browder* (Fayetteville, N. C.).

Gårding, Lars. On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators. Math. Scand. 1, 237–255 (1953).

Considérons un opérateur linéaire aux dérivées partielles d'ordre $2m$,

$$(*)\qquad \sum_{\Sigma\alpha_i\leqq 2m} a_{\alpha_1\cdots\alpha_n}(x_1,\cdots,x_n)\frac{\partial^{\Sigma\alpha_i}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}+t,$$

dont les coefficients sont indéfiniment dérivables. Admettons que cet opérateur est elliptique dans un ouvert S borné, i.e. que la forme

$$a_0(x,\xi)\equiv\sum_{(\Sigma\alpha_i=2m)} a_{\alpha_1,\cdots,\alpha_n}(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$$

est définie positive pour tout $x\in S$. Si t est suffisamment grand, l'opérateur (*) supposé self-adjoint, admet dans S et pour les conditions de Dirichlet généralisées (consistant en l'annulation à la frontière, en un certain sens quadratique, des dérivées de u d'ordre $\leqq m$) une infinité discrète de valeurs propres λ_1+t, λ_2+t, $\cdots$, auxquelles correspondent des fonctions propres orthogonales et normées φ_1, φ_2, $\cdots$, formant un système complet dans $L_2(S)$.

L'auteur établit les deux formules suivantes, relatives à la distribution asymptotique de ces valeurs et fonctions propres:

$$N(t)\equiv\sum_{\lambda_i\leqq t}1=(2\pi)^{-n}w_a(S)t^{n/2m}[1+o(1)],$$

$$\lim_{N\to\infty} N^{-1}\sum_{j=1}^{N}\overline{\varphi_j(x)}\,\varphi_j(y)=\delta_{xy}w_a(x)/w_a(S),$$

ou

$$w_a(x)=\int_{a_0(x,\xi)<1}d\xi,\quad w_a(S)=\int_S w_a(x)dx.$$

La méthode est celle de T. Carleman [Åttonde Skandinaviska Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44; cf. aussi Å. Pleijel, Ark. Mat. Astr. Fys. **27A**, no. 13 (1940); ces Rev. **2**, 291].

Pour $2m>n$, ces formules ont été établies par l'auteur [Kungl. Fysiogr. Sällsk. i Lund Förh. **21**, no. 11 (1951); ces Rev. **14**, 653; et l'oeuvre analysé ci-dessus] et par F. E. Browder, [C. R. Acad. Sci. Paris **236**, 2140–2142 (1953); ces Rev. **15**, 320]. Le présent travail contient la démonstration dans le cas général. *H. G. Garnir* (Liège).

Slobodeckiĭ, L. N. On strongly elliptic differential operators. Doklady Akad. Nauk SSSR (N.S.) **89**, 13–15 (1953). (Russian)

Let L be a system of differential operators of order $2m$ on a domain D of Euclidean n-space,

$$Lu=(-1)^m\sum_{(k)}A^{(k_1,\cdots,k_{2m})}(x)\frac{\partial^{2m}u(x)}{\partial x_{k_1}\cdots\partial x_{k_{2m}}}+Tu$$

acting on N-vector functions u, with the A's $N\times N$ matrices and T of order $<2m$. L is strongly elliptic if

$$(\sum_{(k)}A^{(k_1,\cdots,k_{2m})}(x)\alpha_{k_1}\cdots\alpha_{k_{2m}}y,y)>0$$

for every real non-zero N-vector α. We may consider L as defining an operator L_0 in the N-vector L^2 space over D whose domain is the set of $2m$-times continuously differentiable functions with compact supports in D. If L has constant coefficients and $T=0$, by a Fourier-integral argument due to Van Hove [Nederl. Akad. Wetensch., Proc. **50**, 18–23 (1947); these Rev. **8**, 522] and Gårding [C. R. Acad. Sci. Paris **230**, 1030–1032 (1950); these Rev. **11**, 521] L_0 is a positive operator, being bounded from below by a constant multiple of the m-Dirichlet norm. The author presents examples with $m=2$, $N=2$; $m=4$, $N=1$ to show that positivity no longer holds for variable coefficients. [See the second preceding review.] *F. Browder.*

Oleĭnik, O. A. On equations of elliptic type degenerating on the boundary of a region. Doklady Akad. Nauk SSSR (N.S.) **87**, 885–888 (1952). (Russian)

Let

$$L(u)=\frac{\partial^2u}{\partial x^2}+y^m\frac{\partial^2u}{\partial y^2}+a(x,y)\frac{\partial u}{\partial y}+b(x,y)\frac{\partial a}{\partial x}+c(x,y)u\qquad(m\geqq 1)$$

be defined on a domain D in the half-plane $y>0$ whose boundary consists of a finite number of segments on the x-axis and a portion Γ in the half-plane $y>0$. If $\{P_i\}$ is the set of points which are the end-points of curves of Γ, each connected portion of $\Gamma\cup\{P_i\}$ is assumed to be twice differentiable in terms of arc length, while a, b, and c are real analytic on $\bar{D}$. The author considers the bounded solutions $u(x,y)$ of $Lu=0$ in D satisfying the boundary condition $\partial u/\partial\gamma+Au=\varphi$ on Γ, where γ is a vector at each point of Γ making an acute angle with the normal, A, φ, and the components of γ are continuously differentiable in s, $A\leqq 0$. A proof is sketched for the existence of solutions if either $c<0$ on $\bar{D}$ or $A<0$ on $\Gamma\cup\{P_i\}$. If the further condition is satisfied that one of the following hold: (1) $m=1$ and $a(x,0)\geqq 1$; (2) $1<m<2$, $a(x,0)>0$; (3) $m\geqq 2$, $a(x,0)\geqq 0$; then the solution is unique provided that the set $\{P_i^*\}$ of points of the P_i at which γ makes an obtuse angle with the positive y-axis is empty. More generally, under the same alternate hypotheses, there exists a unique solution of the boundary-value problem taking on prescribed values at the points of $\{P_i^*\}$. *F. Browder* (Fayetteville, N. C.).

Mihlin, S. G. On applicability of a variational method to certain degenerate elliptic equations. Doklady Akad. Nauk SSSR (N.S.) **91**, 723–726 (1953). (Russian)

Consider a plane open set Ω lying in the half-plane $y>0$, whose boundary consists of a finite closed interval $(a,b)=\Gamma'$ of the x-axis and of a curve Γ lying in $y>0$ save for its end-points at a and b. Let M_1 and M_2 be two sets of functions which are twice continuously differentiable on the closed set $\bar{\Omega}=\Omega+\Gamma+\Gamma'$ and vanish on Γ. The functions in M_1 also

vanish on Γ' while those of M_2 have their y derivatives equal to zero on Γ'. Suppose the function $f(y)$ is continuous on the interval $0 \leqq y \leqq Y$, where Y is greater than or equal to the maximum ordinate of the points of the curve Γ; that $f(0)=0$; and $f(y)>0$ when $y>0$. The differential operator $-f(y)\partial^2u/\partial x^2 - \partial^2u/\partial y^2$, considered over the sets of functions M_1 and M_2, produces two operators, which shall be denoted by A_1 and A_2, respectively. The first theorem states that A_1 and A_2 are both positive definite over the Hilbert space $L_2(\Omega)$ (i. e., there exist positive constants γ_i such that $\gamma_i \cdot (u, u) \leqq (A_i u, u)$ for $i=1, 2$ and any u in $L_2(\Omega)$). This fact permits the application of variational methods [cf. Mihlin, Problem of the minimum of a quadratic functional, Gostehizdat, Moscow-Leningrad, 1952; these Rev. **16**, 41] to the boundary-value problems consisting of the partial differential equation $-f(y)\partial^2u/\partial x^2 - \partial^2u/\partial y^2 = \varphi(x, y)$, with φ in $L_2(\Omega)$, subject to either the boundary conditions $u|_\Gamma=0$, $u|_{\Gamma'}=0$; or $u|_\Gamma=0$, $u_y|_{\Gamma'}=0$; and also to the corresponding boundary-value problems with the homogeneous equation and non-homogeneous boundary conditions. Further results concern the case when $f(y)=y^\alpha\omega(y)$, where $\omega(y) \geqq k > 0$ for $0 \leqq y \leqq Y$ and $\alpha>0$, and the differential operator is

$$-\frac{\partial^2u}{\partial x^2} - \frac{\partial}{\partial y}\left(f(y)\frac{\partial u}{\partial y}\right).$$

J. B. Diaz (College Park, Md.).

Mihlin, S. G. On the theory of degenerate elliptic equations. Doklady Akad. Nauk SSSR (N.S.) **94**, 183–185 (1954). (Russian)

The author considers the differential equation

$$L(u) = -\sum_{i,k=1}^{m} \frac{\partial}{\partial x_i}\left(A_{ik}\frac{\partial u}{\partial x_k}\right) = f(x),$$

which is of elliptic type in a bounded domain Ω of Euclidean space; the coefficients A_{ik} are supposed sufficiently regular. Ω is supposed to be the sum of a finite number of star-shaped domains, to permit the applicability of the inclusion theorem of S. L. Sobolev [Some applications of functional analysis to mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; these Rev. **14**, 565]. The degeneracy of L consists in the existence of a proper subset of the boundary Γ of Ω such that on this subset some of the eigenvalues of the matrix $\|A_{ik}\|$ are zero. The present note is concerned with the formulation of spectral properties of the operator L under various boundary conditions and two types of degeneracy. Some of the results have already been announced for $m=2$ in the paper reviewed above. *J. B. Diaz.*

Morrey, Charles B., Jr. Second order elliptic systems of differential equations. Proc. Nat. Acad. Sci. U. S. A. **39**, 201–206 (1953).

This note describes results obtained by the author concerning second-order linear elliptic systems of differential equations of the form

$$(2) \qquad a_{ij}^{\alpha\beta}u^j{}_{x_\alpha x_\beta} = c^\alpha{}_{ij}u^j{}_{x_\alpha} + d_{ij}u^j + f_i$$

(summation convention, $1 \leqq i, j \leqq N$, $1 \leqq \alpha, \beta \leqq \nu$), and more generally

$$(1) \qquad \int_{R^*} (a_{ij}^{\alpha\beta}u^j_{x_\beta} + b^\alpha{}_{ij}u^j + e_i^\alpha)dx_\alpha' = \int_R (c^\alpha{}_{ij}u^j{}_{x_\alpha} + d_{ij}u^j + f_i)dx,$$

where R is any ν-cell in a domain G, R^* its boundary, $dx_\alpha' = (\zeta_\alpha, n)dS$, where ζ_α is the unit vector in the x_α-direction, n is the exterior normal, dS the differential of $(\nu-1)$-area on R^*. Weak solutions are considered for both (1) and (2) with very mild restrictions on the coefficients. Using a technique like that employed by Schauder [Math. Z. **38**, 257–282 (1934)] for the single second-order equation, F. John's method of constructing fundamental solutions for elliptic differential operators [Comm. Pure Appl. Math. **3**, 273–304 (1950); these Rev. **13**, 40] and the author's own technique of obtaining and applying Dirichlet growth estimates to obtain Hölder continuity of derivatives of solutions of elliptic equations, the author obtains extremely powerful results on the regularity properties of solutions of both linear and non-linear elliptic systems of the second-order as well as regular multiple-integral variational problems.

Worthy of specific mention are the following results. (1) If z is of class C^2 and is the solution on G of a general elliptic second-order system

$$\varphi_i(x, z, z_{x_\alpha}, z_{x_\alpha x_\beta}) = 0$$

in which the φ_i are of class C^1, then the second derivatives of z satisfy a μ-Hölder condition on every compact subdomain of G for any $0<\mu<1$. If the derivatives of the φ_i satisfy a μ-Hölder condition, then the derivatives of z up to order $n+2$ satisfy a μ-Hölder condition on every compact subdomain of G. (2) If z is of class C^1 and is the solution of a regular variational problem with integrand f, where f and its derivatives with respect to the $p_\alpha{}^i$ are of class C^1 in all arguments, then z and its first derivatives satisfy a μ-Hölder condition on all compact subdomains for any $0<\mu<1$. If f and $f_{p_\alpha{}^i}$ have all their derivatives through order n satisfying a μ-Hölder condition, the same is true up to the $(n+1)$th derivatives of z on compact subdomains of G.

Existence theorems are given for systems of the type of (1) and (2) and results on the regularity of solutions on the boundary under the additional assumptions of strong ellipticity and smoothness of boundary and boundary-values. *F. Browder* (Fayetteville, N. C.).

Nirenberg, Louis. On nonlinear elliptic partial differential equations and Hölder continuity. Comm. Pure Appl. Math. **6**, 103–156; addendum, 395 (1953).

Let $z(x, y)$ be a solution of the general non-linear elliptic partial differential equation of the second order in the plane, $F(x, y, z, p, q, r, s, t)=0$, $4F_rF_t - F_s{}^2 > 0$. It is shown in this paper that if F has continuous first partial derivatives, then every solution z of class C^2 has its second derivatives satisfying a Hölder condition on every compact subdomain of its domain of regularity. Further, Hölder differentiability of F implies Hölder differentiability of the solution z and its derivatives [cf. the paper of Morrey reviewed above]. All the results are derived from the fundamental result on a pair of functions $p(x, y)$, $q(x, y)$, bounded by a constant K with continuous first partial derivatives satisfying the inequality (extended quasi-conformality)

$$p_x{}^2 + p_y{}^2 + q_x{}^2 + q_y{}^2 \leqq k(p_yq_x - p_xq_y) + k_1; \quad k, k_1 \geqq 0.$$

The fact that such functions p and q satisfy a Hölder condition on every compact subdomain with constants depending only on k, k_1, K and the distance from the boundary, is established using Dirichlet growth estimates of a type slightly varying from those introduced by Morrey [Trans. Amer. Math. Soc. **43**, 126–166 (1938); Univ. California Publ. Math. **1**, 1–130 (1943); these Rev. **6**, 180]. A sharpening of the reasoning to deal with the boundary behaviour enables the author to reprove the Leray-Schauder theorem for the existence of solutions for the Dirichlet problem for

the general homogeneous quasi-linear elliptic equation on a convex plane domain [Ann. Sci. Ecole Norm. Sup. (3) **51**, 45–78 (1934)] using only the Schauder fixed-point theorem.
F. Browder (Fayetteville, N. C.).

★Brousse, P., et Poncin, H. Quelques résultats généraux concernant la détermination de solutions d'équations elliptiques par les conditions aux frontières. Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 17–24. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

The authors discuss the Dirichlet problem for the equation

$$(*)\qquad V_{xx}+V_{yy}+a_1y^{-1}V_y+a_2y^{-2}V+a_3=0$$

(a_1, a_2, a_3 constants) in the case when the boundary of the domain contains a segment of the x-axis. If $a_2=a_3=0$, $a_1<0$, the Dirichlet problem is solvable uniquely. If $a_2=a_3=0$, $a_1-2>0$, the problem is in general not solvable. Additional results are obtained for unbounded domains and for the case $a_1=a_3=0$, $a_2<0$. *M. H. Protter* (Berkeley, Calif.).

Hyman, Morton A. Concerning analytic solutions of the generalized potential equation. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 408–413 (1954).

It is shown that any solution of $F_{xx}+F_{yy}+py^{-1}F_y=0$ (p a real constant $\neq 0, -1, -2, \cdots$), analytic in a region R containing a segment of the x-axis is determined throughout R by its values $F_0(x)$ on the segment of the x-axis contained in R. This generalizes a result of Weinstein [Trans. Amer. Math. Soc. **63**, 342–354 (1949); these Rev. **9**, 584].
M. H. Protter (Berkeley, Calif.).

Weinstock, Robert. Inequalities for a classical eigenvalue problem. J. Rational Mech. Anal. **3**, 745–753 (1954).

Soit D un domaine simplement connexe du plan des x, y, de frontière C, courbe fermée simple. Problème de Stekloff [Ann. Sci. Ecole Norm. Sup. (3) **19**, 455–490 (1902)]: trouver ϕ deux fois continûment différentiable dans D (et alors indéfiniment différentiable) solution de (1) $\nabla^2\phi=0$ (∇^2=Laplacien), avec (2) $\partial\phi/\partial n=h\phi$, h=constante, n=normale extérieure à C. Ce problème n'admet de solution non nulle que pour un ensemble dénombrable de valeurs de h (valeurs propres): $0=h_0<h_1\leqq h_2\leqq\cdots$ [cf. Courant et Hilbert, Methoden der mathematischen Physik, t. 1, 2ème éd., Springer, Berlin, 1931, pp. 340, 341, 400, 401]. On va donner deux majorations de h_1: (A) on suppose C analytique; soit L sa longueur. Alors (3) $h_1\leqq 2\pi/L$ et il n'y a égalité que si C est un cercle. (B) C est régulière par morceaux. On prend pour origine le centroide de C (i.e. $\int_C xds=\int_C yds=0$). Soit A l'aire de D et $J=\int_C(x^2+y^2)ds$. Alors (4) $h_1\leqq 2AJ^{-1}$ et il n'y a égalité que si C est un cercle.

Principe des démonstrations: on a dans tous les cas:

$$(5)\qquad h=\min_u\left(\left(\int_C u^2ds\right)^{-1}\iint_D(u_x^2+u_y^2)\,dxdy\right),$$

$u\in\Omega$=classe des fonctions une fois continûment dérivables dans D et telles que $\int_C uds=0$. On applique (5) à $u(x)=x$ et $u(x)=y$ d'où l'on déduit (4). La démonstration de (3) est plus délicate: on montre, par adaptation d'une méthode de Szegö [J. Rational Mech. Anal. **3**, 343–356 (1954); ces Rev. **15**, 877] qu'il existe une fonction $g(z)=U(x,y)+iV(x,y)$, analytique univalente dans D, d'image le disque unité, telle que $\int_C Uds=0$ et $\int_C Vds=0$; donc U et V sont dans Ω; on leur applique (5), d'où l'on déduit le résultat. Comparaison de (A) et (B): si D est convexe et C de courbure continue par morceaux, alors [Weinstock, Department of Math., Stanford Univ., Tech. Rep. **37** (1954)] $2AJ^{-1}<2\pi L^{-1}$. L'auteur donne un exemple où l'inégalité contraire a lieu (D non convexe).

Généralisation de (A): l'auteur remplace (2) par (2 bis) $\partial\phi/\partial n=hp\phi$, p fonction continue positive sur C; on a un résultat analogue à (3): (3 bis) $h_1\leqq 2\pi/L'$, $L'=\int_C p(s)ds$.

Remarque: (B) se généralise à R^n (même méthode).
J. L. Lions (Nancy).

Fer, Francis. Construction d'une solution à singularité mobile de l'équation $\square u-ku=0$. C. R. Acad. Sci. Paris **239**, 1191–1192 (1954).

Aržanyh, I. S. Representation of a displacement vector by retarded potentials. Doklady Akad. Nauk SSSR (N.S.) **94**, 393–396 (1954). (Russian)

The system of equations governing the dynamics of a three-dimensional elastic body is

$$(*)\qquad \Delta_q{}^t\mathbf{u}=\mathbf{f},$$

where the differential operator is Lamé's operator:

$$\Delta_q{}^t=\alpha\,\mathrm{grad}_q\,\mathrm{div}_q-\beta\,\mathrm{rot}_q\,\mathrm{rot}_q-\frac{\partial^2}{\partial t^2}.$$

The boundary-value problem of the first kind requires a solution of (*), the values $\mathbf{u}_s$ of the vector $\mathbf{u}$ being prescribed on the boundary surface S, while in the second boundary-value problem the values of

$$d_n\mathbf{u}=\lim_{q\to S}\{c(\mathbf{n}\nabla_q)\mathbf{u}+(\alpha-c)\mathbf{n}\,\mathrm{div}_q\,\mathbf{u}-(\beta-c)[\mathbf{n}\,\mathrm{rot}_q\,\mathbf{u}]\}$$

are prescribed on the boundary. There exists a great analogy between the theory of the system (*) and that of the wave equation in three dimensions. A significant role in the theory of the wave equation is played by Kirchhoff's formula. The author now develops a formula for the Lamé operator which may be said to be of Kirchhoff's type in that it expresses an arbitrary vector $\mathbf{u}$ in terms of its "retarded" values of $\Delta_q{}^t\mathbf{u}$, $d_n\mathbf{u}$ and $\mathbf{u}_s$. The analysis leads to the consideration of retarded vector potentials which appear as natural generalizations of the customary retarded potentials occurring in the theory of the wave equation. *J. B. Diaz.*

★Nehari, Zeev. On the biharmonic Green's function. Studies in mathematics and mechanics presented to Richard von Mises, pp. 111–117. Academic Press Inc., New York, 1954. $9.00.

Let $g(P,Q)$ be the biharmonic Green's function of a domain D in the plane with a smooth boundary B, corresponding to vanishing boundary values and normal derivatives. Hadamard's conjecture that $g(P,Q)$ is always positive was disproved in several papers [see R. J. Duffin, J. Math. Physics **27**, 253–258 (1949); these Rev. **10**, 534; P. R. Garabedian, Pacific J. Math. **1**, 485–524 (1951); these Rev. **13**, 735; Ch. Loewner, ibid. **3**, 417–436 (1953); G. Szegö, ibid. **3**, 437–446 (1953); these Rev. **14**, 1085]. It is shown by the author that there exist nevertheless subdomains of D characterized by simple geometrical properties in which $g(P,Q)$ is positive. He proves that $g(P,Q)$ is positive if P lies in a circle of radius r around Q such that B lies between concentric circles of radii $2r$ and $5r$. A still simpler result is obtained if the Green's function $G(P,Q)$ of a spatial domain with the same boundary conditions is considered. The above assertion holds already if the boundary lies outside of the circle of radius $2r$. The paper contains further interesting

inequalities giving upper bounds for $G(P, Q)$ if a finite number of points on the boundary are known. An essential tool in the proofs is the fact that the quadratic functionals $\int\int g(P, Q)f(P)f(Q)\,dPdQ$ and $\int\int G(P, Q)f(P)f(Q)\,dPdQ$ are positive and monotonically increasing with the domain.

C. Loewner (Stanford, Calif.).

Kikuta, Takashi. Super-stationary variational method. Progress Theoret. Physics **12**, 10–16 (1954).

If $\{\phi_k\}$ is a set of trial functions close to the correct eigenfunctions $\{\psi_k\}$ of an eigenvalue equation, $A\psi_k=\lambda_k B\psi_k$, the author shows that the set, $\dot\phi_k=\phi_k-\sum_{k\neq k'}\alpha_{kk'}\phi_{k'}$ where

$$\alpha_{kk'}=[\phi_k(A-w_kB)\phi_{k'}](w_k-w_{k'})^{-1}$$

and

$$w_k=(\phi_kA\phi_k)(\phi_kB\phi_k)^{-1},$$

provides an even closer set of trial functions. In fact, if $\phi_k=(1+\Delta)\psi_k$, w_k, calculated using $\dot\phi_k$, reproduces the eigenvalue λ_k to the accuracy $\lambda_k+O(\Delta^4)$. Applications of the new variational method are made to scattering and bound state problems.

A. Salam (Cambridge, England).

Karmanov, V. G. On a boundary problem for an equation of mixed type. Doklady Akad. Nauk SSSR (N.S.) **95**, 439–442 (1954). (Russian)

For the equation

$$(*)\qquad \frac{\partial^2 u}{\partial x^2}+\operatorname{sgn} y\frac{\partial^2 u}{\partial y^2}=0$$

the following problem is discussed: Let G be the domain (m-ply connected) bounded in the upper half-plane by the arcs $\gamma_1, \gamma_2, \cdots, \gamma_m$ with end-points on the x-axis at the points $A_1, B_1; A_2, B_2; \cdots; A_m, B_m$. The segments $A_i, B_i, i=2, \cdots, m$, are non-overlapping and contained in the interval A_1, B_1. In the lower half-plane G is bounded by the line segments (A_i, C_i) and (B_i, C_i), where the C_i are the points of intersection of the characteristics of (*) given by $y=A_i-x$ and $y=x-B_i$. The boundary-value problem concerns the existence and uniqueness of a solution of (*) in G subject to the conditions $u=f_i$ on γ_i, $i=1, 2, \cdots, m$, $u=\psi_1$ on (A_1, C_1), $u=\psi_i$ on (B_i, C_i), $i=2, \cdots, m$. The functions f_i, ψ_i are prescribed, the f_i being continuous and the ψ_i twice differentiable.

M. H. Protter (Berkeley, Calif.).

***Karmanov, V. G. On a boundary problem for equations of mixed type.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 7 pp. (mimeographed). \$3.00.

Translation of the paper reviewed above.

Diaz, J. B., and Ludford, G. S. S. On two methods of generating solutions of linear partial differential equations by means of definite integrals. Quart. Appl. Math. **12**, 422–427 (1955).

On considère l'équation linéaire hyperbolique sous forme canonique:

$$(1)\qquad L(u)=u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=0.$$

Soit $U(x, y, \alpha)$ une famille de solutions de (1) dépendant du paramètre α. Si f est une fonction de α telle que l'intégrale

$$(2)\qquad u(x, y)=\int_{\alpha_1}^{x} f(\alpha)U(x, y, \alpha)d\alpha$$

ait un sens, et qu'on puisse dériver sous le signe somme jusqu'à $\partial^2/\partial x\partial y$, et si, pour $x=\alpha$, on a: $\partial U/\partial y+a(x, y)U=0$, alors (2) est solution de (1) [cf. Le Roux, Ann. Sci. Ecole Norm. Sup. (3) **12**, 227–316 (1895)].

D'un autre coté, par généralisation d'une idée de Bergmann [NACA Tech. Note no. **972** (1945), pp. 34, 35; Amer. J. Math. **74**, 444–474 (1952); ces Rev. **7**, 342; **15**, 229] les auteurs montrent ceci (par dérivation sous le signe somme): soit $E(x, y, t)$ solution de

$$(1-t^2)(E_{yt}+aE_t)-t^{-1}(E_y+aE)+2xtL(E)=0,$$

telle que pour $x\neq 0$, $(1-t^2)^{1/2}(xt)^{-1}(E_y+aE)$ soit continue pour $t=0$, et nulle pour $t=\pm 1$. Alors si f est quelconque, une fois continuement différentiable, la fonction u définie par

$$u(x, y)=\int_{-1}^{1} E(x, y, t)f(\tfrac{1}{2}x(1-t^2))(1-t^2)^{-1/2}dt$$

est solution de (1).

Les auteurs montrent que (3) se ramène essentiellement à une solution du type (2). Applications à la dynamique des fluides.

J. L. Lions (Nancy).

***Arf, C. On a generalization of Green's formula and its application to the Cauchy problem for a hyperbolic equation.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 69–78. Academic Press Inc., New York, 1954. \$9.00.

L'auteur généralise la formule de Green pour un opérateur linéaire F aux dérivées partielles du second ordre quelconque. Il utilise pour cela au lieu d'une fonction R un opérateur différentiel $\mathfrak{R}$ qui le conduit à la formule intégrale:

$$(1)\qquad \int_D \Omega(\mathfrak{R}F\varphi-\varphi F^*R)=\int_B\sum_{i=1}^{n}\Omega_i(\mathfrak{R}_iF+L_i)\varphi.$$

L'auteur cherche les relations que doivent vérifier les opérateurs $\mathfrak{R}$, L_i et la fonction g pour que la quantité à intégrer au deuxième membre s'annulle sur une portion de la frontière B du domaine D représentée par $g=0$.

Application au problème de Cauchy pour l'équation des ondes à un nombre pair de variables: $g=0$ étant l'équation du cône caractéristique, l'auteur détermine les opérateurs $\mathfrak{R}_i$ à l'aide d'un système d'èquations différentielles. Par un choix simple dés opérateurs L_i il déduit de la formule (1) l'expression, sans intégrales singulières, de la solution du problème de Cauchy pour l'équation $F\varphi=\Phi$ quand F est l'opérateur des ondes.

Y. Fourès-Bruhat (Marseille).

Durand, Emile. Recherche des solutions de l'équation des ondes planes.

$$\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}-K_0{}^2\psi=f(x, vt).$$

Ann. Fac. Sci. Univ. Toulouse (4) 17 (1953), 229–264 (1954).

The main result of this paper is as follows: Let C be a closed curve. Let S be the region cut out of the area bounded by C by that angle between the lines $\tau-t=\pm(\xi-x)$ in the (ξ, τ)-plane which lies in $\tau\leqq t$. Then

$$(*)\qquad \tfrac{1}{2}v\left[\frac{1}{v^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-k_0{}^2\right]\int_S\int\psi(\xi, v\tau)I_0(k_0\gamma)d\xi d\tau=\epsilon\psi(x, vt),$$

where $\epsilon=0$, $\frac{1}{2}$ or 1 according as (x, t) lies outside, on or within C, where I_0 is the Bessel function of imaginary argument and where $\gamma^2=v^2(t-\tau)^2-(x-\xi)^2$. This result is used to obtain solutions of the differential equation in the title under various boundary conditions. The proof of (*) is

rather involved; and (*) is not necessarily true when (x, t) is on C. It is true only if C touches neither characteristic at (x, t). *E. T. Copson* (St. Andrews).

Pastori, Maria. Fronti d'onda ed equazioni tensoriali. Matematiche, Catania **8**, no. 2, 28–42 (1953).

An expository paper on the role of characteristic manifolds and wave fronts for some of the tensor hyperbolic systems of partial differential equations of mathematical physics. *C. R. De Prima* (Pasadena, Calif.).

Krzyżański, Mirosław. Sur le problème de Fourier dans une région indéfinie. Arch. Mech. Stos. **5** (1953), 584–588 (1954). (Polish. Russian and French summaries)

The author considers solutions of the class C^2 of the partial differential equation of parabolic type,

$$u_{xx}+a(x,t)u_x-u_t=0,$$

in an infinite strip $[0\leq x\leq k, -\infty<t<\infty]$, where solutions satisfy the conditions $u(0,t)=\psi_1(t)$, $u(k,t)=\psi_2(t)$. In general this problem has infinitely many solutions. However, if one requires that the solution $u(x,t)$ satisfies the inequality $|u(x,t)|\leq M\exp(\lambda(k)(t^2+1)^{1/2})$, where M and $\lambda(k)<(\pi/2k)^2$ are conveniently chosen constants, then, as the author shows, the solution is unique. Using results of his previous investigations, the author indicates that certain approximation methods can be used to obtain the solution of the problem mentioned above. An analogous procedure can be used to solve a similar problem for the equations in three variables, $u_{xx}+u_{yy}+au_x-u_t=0$. *S. Bergman.*

O'Sullivan, D. G. A transformation of solutions of diffusion equations valid for certain initial and boundary conditions. Experientia **10**, 455–456 (1954).

Many mathematical solutions are available for the two-region diffusion problem in which one region initially possesses a uniform concentration and the other a zero concentration of the diffusing substance. This paper gives a method of transformation of these solutions into solutions of more difficult problems in which the initial concentrations in both regions are zero but the diffusing substance is produced at a constant rate in one of the regions. These solutions will apply only for certain initial and boundary conditions. An illustration is given of the application of this transformation to linear diffusion in the normal direction across a plane interface. *C. G. Maple* (Ames, Iowa).

***Finkel'štein, B. N., and Halatnikov, I. M. Cooling of a cylinder in a well stirred fluid.** Sbornik posvyaščennyĭ semidesyatiletiyu akademika A. F. Ioffe [Collection in honor of the seventieth birthday of academician A. F. Ioffe], pp. 105–108. Izdat. Akad. Nauk SSSR, Moscow, 1950.

Lochs, Gustav. Die Diffusion aus einer Platte oder Kugel bei geringem Umsatz. Z. Angew. Math. Mech. **34**, 79–80 (1954).

Let $u=(C_A-\bar{C})/C_A$, where C_A and $\bar{C}$ are respectively the initial and time-averaged concentration of a gas in a solid body. In general, for a disc-like body u may be represented in the form

$$1-u=\frac{8}{\pi^2}\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+1)^2}\exp\left\{-(2\nu+1)^2\frac{\pi^2}{a^2}Dt\right\},$$

where D is the coefficient of diffusion and a is the density of the material. By a substitution on the independent variable it is shown that

$$u=\frac{4}{a}\left(\frac{Dt}{\pi}\right)^{1/2}-R_1,\quad 0\leq R_1\leq\frac{16}{a^3}(\pi(Dt)^3)^{1/2}\exp\left\{-\frac{a^2}{4Dt}\right\}.$$

This may be used to calculate approximate values for u. *C. G. Maple* (Ames, Iowa).

Itô, Seizô. The fundamental solution of the parabolic equation in a differentiable manifold. II. Osaka Math. J. **6**, 167–185 (1954).

The author treats the differential equation

$$a^{ij}(t,x)\frac{\partial^2}{\partial x^i\partial x^j}+b^i(t,x)\frac{\partial}{\partial x^i}+c(t,x)-\frac{\partial}{\partial t}=0$$

for x in a domain with boundary (whose closure is compact) in a general m-dimensional manifold and for t in an interval $s_0<t<t_0$. The equation is not assumed to be formally self-adjoint inside the domain, but is assumed so on the boundary, and the boundary condition imposed is mixed Dirichlet-Neumann. Subject to differentiability and Hölder assumptions, he establishes the existence of a fundamental solution of the equation and of its adjoint, in close analogy to results previously obtained by himself for a non-compact space but without boundary conditions [same J. **5**, 75–92 (1953); these Rev. **15**, 36]. *S. Bochner* (Princeton, N. J.).

Yosida, Kôsaku. On the integration of the temporally inhomogeneous diffusion equation in a Riemannian space. Proc. Japan Acad. **30**, 19–23 (1954).

Let R be a connected domain of an m-dimensional, orientable C^∞ Riemann space and let $L(R)$ be the Banach space of Borel measurable functions integrable over R. The author has established the existence of a solution to the initial-value problem associated with the temporally inhomogeneous diffusion equation with C^∞ coefficients: $\partial f(t,s,x)/\partial t=A_{ts}f(t,s,x)$, $\lim_{t\searrow s}f(t,s,x)=f(x)\in L(R)$ for almost all x; here A_{ts} is assumed to be an elliptic differential operator of second order in x. Denoting the class of all C^∞ functions with compact carriers by $D(R)$, the operator A_{ts} is first defined on $D(R)$ to $L(R)$. Writing $\bar{A}_{ts}$ for the least closed extension of A_{ts}, the author assumes for the resolvent that $nR(n;\bar{A}_{ts})$ is a transition operator for n sufficiently large (independent of t), defining a strongly continuous function of t for each n. With this hypothesis he is able to find a solution $f(t,s,x)\in L(R)$. After solving the initial-value problem associated with the approximating operator $n\bar{A}_{ts}R(n;\bar{A}_{ts})$, the author obtains a weak solution to the given problem by a limiting process (on n) and then makes use of a parametrix [see K. Yosida, Osaka Math. J. **5**, 65–74 (1953); these Rev. **15**, 36] to show that this weak solution is equivalent to a genuine solution. *R. S. Phillips.*

Yosida, Kôsaku. On the integration of the temporally inhomogeneous diffusion equation in a Riemannian space. II. Proc. Japan Acad. **30**, 273–275 (1954).

The purpose of the present note is to show that the existence proof developed in the paper reviewed above may be modified so as to yield the existence of a solution admitting of a kernel representation:

$$f(t,s,x)=\int_R P(t,s,x,y)f(y)\,dy$$

for each $f\in L(R)$. The kernel function $P(t,s,x,y)$ can be chosen so that it satisfies the equation $\partial P/\partial t=A_{ts}P$ for $t>s$. *R. S. Phillips* (Los Angeles, Calif.).

Mendes, M. Sur une équation aux dérivées partielles du troisième ordre. Ann. Fac. Sci. Univ. Toulouse (4) **17** (1953), 97–137 (1954).

In part one of the paper several types of transformations are applied to equations of the form

$$(*)\qquad \frac{\partial^3 u}{\partial x\partial y\partial z}+a(x)\frac{\partial^2 u}{\partial y\partial z}+d(x,y)\frac{\partial u}{\partial x}+g(x,y,z)u=0,$$

and under each the coefficient d of u_x and the combination of coefficients $H\equiv ad-g$ are shown to remain invariant. Using special forms of H, such as $H=0$, $H=f(x)$, $\cdots$, a classification of types or classes of equations (*) is made. In the second part of the paper various forms of H and solutions of the equation adjoint to (*) are shown to play important roles in reducing the above equation to simpler forms. In part three, several boundary-value problems are treated, two of which we note here. (1) Find a solution of (*) taking on preassigned values on three mutually perpendicular planes. (2) Show there exists an infinite number of solutions of (*) each of which on the cylinder $y=f(x)$, along with their first derivatives with respect to x, takes on preassigned functions of x and y. [Some of these results have been announced earlier by the author in C. R. Acad. Sci. Paris **225**, 619–620 (1947); these Rev. **9**, 147.]

F. G. Dressel (Durham, N. C.).

Difference Equations, Special Functional Equations

Cooke, K. L. The rate of increase of real continuous solutions of algebraic differential-difference equations of the first order. Pacific J. Math. **4**, 483–501 (1954).

Studied is the rate of increase, for $t\to+\infty$, of real solutions of the equation

$$(1)\qquad P(t,u(t),u^{(1)}(t),u(t+1),u^{(1)}(t+1))=0,$$

where $P(t,u,v,\cdots)$ is a real polynomial in $t,u,v,\cdots$. The developments presented constitute a confirmation of the work in this field of E. Borel, O. E. Lancaster, S. M. Shah, T. Vijayaraghavan and others. A solution $u(t)$ is proper if t_0 exists such that $u(t)$ satisfies (1) for $t\geqq t_0$ and $u^{(1)}(t)$ is continuous for $t\geqq t_0$. Some of the results are as follows. If $g(t)$ is any increasing function such that $g(t)\to+\infty$ for $t\to+\infty$, one can construct an equation (1), which has a proper solution $u(t)\geqq g(t)$; one can construct an equation $P(t,u(t),u^{(1)}(t+1))=0$, having a proper solution $u(t)\geqq g(t)$ for $t=t_n$, $t_n\to+\infty$ with n. Given equation (1), without $u^{(1)}(t+1)$, there exists an $A>0$ such that to each proper non-decreasing or non-increasing solution $u(t)$ there correspond t_n, $t_n\to+\infty$, such that (2) $|u_n(t_n)|<e_2(At_n)$. Given an equation $P(t,u^{(1)}(t),u(t+1))=0$, and $A>0$ exists so that to every proper solution $u(t)$ there correspond t_n ($t_n\to+\infty$) for which (2) holds. *W. J. Trjitzinsky.*

Mirolyubov, A. A. The solution of a class of linear differential-difference equations. Mat. Sbornik N.S. **34(76)**, 357–384 (1954). (Russian)

This results of this paper, here given with detailed proofs, were announced in an earlier paper [Doklady Akad. Nauk SSSR (N.S.) **85**, 1209–1210 (1952); these Rev. **14**, 285].

J. M. Danskin (Washington, D. C.).

Hosszú, Miklós. On the functional equation of transitivity. Acta Sci. Math. Szeged **15**, 203–208 (1954).

The operation $z=x*y$ is said to be transitive provided $(x*t)*(y*t)=x*y$. The author investigates the algebraic properties of such an operation and applies them to solve the functional equation $F[F(x,t),F(y,t)]=F(x,y)$ in the class of continuous and strictly monotonic functions of two real variables. Adding the hypothesis of differentiability, he also solves $F[G(x,t),H(y,t)]=K(x,y)$ by reducing its solution to the solution of a differential equation.

E. F. Beckenbach (Los Angeles, Calif.).

Integral Equations

***Gloden, A. Résolution de l'équation intégrale de Fredholm de seconde espèce dans le cas des noyaux d'ordre fini. (Introduction à l'étude des équations intégrales.)** Luxembourg, 1954. 30 pp. 100 francs belges (chez l'auteur, 11 rue Jean Jaurès, Luxembourg).

on order

The pamphlet gives the terminology of linear integral equations and works out (with examples) the solution of the Fredholm equation of the second kind in the case of a degenerate kernel (or kernel of finite order), i.e., $K(x,y)=\sum_{i=1}^n u_i(x)v_i(y)$. It stops short of giving the solution of the equation $g(x)=f(x)-\lambda\int_a^b K(x,y)f(y)dy$ for this case in the form $f(x)=g(x)-\lambda\int_a^b[D_1(x,y;\lambda)/D(\lambda)]g(y)\,dy$ and treats only the solutions of the homogeneous equation when $D(\lambda)=0$. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Tautz, G. L. Über die Existenz von Eigenwerten. Arch. Math. **5**, 401–413 (1954).

The paper begins with a new proof of the existence of characteristic values for a real symmetric kernel, or for a completely continuous symmetric operator in a real Hilbert space; the proof is based on E. Schmidt's well known dissection of the kernel into two parts, one of finite rank and one of small norm.

The author then purports to prove that if a kernel $K_0(x,y)$ has a real characteristic value, and $\mathfrak{K}$ is a compact set of kernels to which K_0 belongs (compactness being with respect to the $\mathfrak{L}^2$ metric, say), then every kernel belonging to $\mathfrak{K}$ and to some neighbourhood of K_0 has a real characteristic value. A counter-example to this statement is provided by taking

$$K_0(x,y)=\varphi_1(x)\varphi_1(y)+\varphi_2(x)\varphi_2(y),$$

where $\|\varphi_1\|=\|\varphi_2\|=1$, $(\varphi_1,\varphi_2)=0$, and considering the neighbouring kernels

$$K_\epsilon(x,y)=K_0(x,y)+\epsilon[\varphi_1(x)\varphi_2(y)-\varphi_2(x)\varphi_1(y)],$$

none of which has a real characteristic value for real nonzero ϵ. *F. Smithies* (Cambridge, England).

Schmeidler, Werner, und Morgenstern, Dietrich. Zur Übertragung des Alternativsatzes der Fredholmschen Theorie auf algebraische Integralgleichungen. Arch. Math. **5**, 452–457 (1954).

Consider the equation

$$(1)\qquad z^n(s)-\mathfrak{K}_n(z,y)-\sum_{\beta=0}^{n-1}\mathfrak{K}_\beta(z,y)\equiv P(z,y)=0,$$

where

$$\mathfrak{K}_\beta(z,y)=\sum_{\alpha+\alpha_1+\cdots+\alpha_\nu=\beta} z^\alpha(s)$$

$$\times\int_0^1\cdots\int_0^1 K_{\alpha\alpha_1\cdots\alpha_\nu}(s,t_1,\cdots,t_\nu)y^{\alpha_1}(t_1)\cdots y^{\alpha_\nu}(t_\nu)dt_1\cdots dt_\nu$$

$$(\beta=0,1,\cdots,n;\ 0\leqq\alpha\leqq\beta).$$

The $K_{\alpha,\alpha_1\cdots\alpha_v}(s, t_1, \cdots, t_v)$ are real and continuous; for $\alpha_1=\alpha_2=\cdots=\alpha_v=0$ they depend only on s, while $K_{n0\cdots 0}=0$. Assumptions. The discriminant $D(y)$ of $P(z, y)$ considered as polynomial in z is $\neq 0$ for all real continuous $y(s)$ (including $y\equiv 0$), and the discriminant $D_0(y)$ of $P_0(z, y)\equiv z^n-\mathfrak{K}_n(z, y)$ is likewise $\neq 0$ for all real continuous y (except $y\equiv 0$). In addition, the equation $P(z, 0)=0$ has at least one real solution $z(s)$.

With these assumptions the following theorem is stated: Concerning the equation

$$(2)\qquad y^n(s)-\mathfrak{K}_n(y, y)=\sum_{\beta=0}^{n-1}\mathfrak{K}_\beta(z, y)$$

only the following two possibilities exist: I) (2) has a (not necessarily real) continuous solution y if the $\mathfrak{K}_\beta$ $(\beta=0, \cdots, n-1)$ satisfy the above assumptions but are arbitrary otherwise; II) (2) with $\mathfrak{K}_\beta=0$ $(\beta=0, 1, \cdots, n-1)$ has a non-zero solution. The theorem is proved under the additional assumption: the equation $(\mu y(s))^n-\mathfrak{K}_n(\mu y, y)=0$ (μ real) has no non-zero solution for $\mu\geqq 1$.

E. H. Rothe (Ann Arbor, Mich.).

Tranter, C. J. A further note on dual integral equations and an application to the diffraction of electromagnetic waves. Quart. J. Mech. Appl. Math. 7, 317–325 (1954).

The author studies the solution of the dual integral equations

$$\int_0^\infty G(u)f(u)J_\nu(\rho u)du=g(\rho),\quad 0<\rho<1;$$

$$\int_0^\infty f(u)J_\nu(\rho u)du=0,\quad \rho>1,$$

where $G(u)$ and $g(\rho)$ are given functions of the indicated variable and ν is now not zero. For a special choice of G, g and ν, this system of equations is equivalent to the diffraction of a plane wave by a narrow slit in an infinite screen and the author discusses the solution. *A. E. Heins.*

Geronimus, Ya. L. On some integral equations. Doklady Akad. Nauk SSSR (N.S.) 98, 5–7 (1954). (Russian)

Considered is the equation (1) $(2\pi)^{-1}\int_C p(s)\log r^{-1}ds=q(\sigma)$, which determines the density $p(s)$ of the logarithmic potential of a simple layer, situated on a smooth curve C of length l, when the value $q(\sigma)$ of the potential is given on C; $r=|z-\zeta|$; $z, \zeta\in C$; s, σ are arc coordinates of z, ζ. Solution of (1) is studied first when $q(\sigma)$ has a derivative with a suitable modulus of continuity. The author also considers the case when the latter condition is replaced by the requirement of mere continuity of $q(\sigma)$. The methods used are from the field of integral equations in the sense of principal values (Mushelišvili and others) and the interest of the paper is in its relation to the contact problem of elasticity. *W. J. Trjitzinsky* (Urbana, Ill.).

Krein, M. G. On integral equations generating differential equations of 2nd order. Doklady Akad. Nauk SSSR (N.S.) 97, 21–24 (1954). (Russian)

Let $H(t)$ $(0\leqq t<2R)$ be a continuous function such that for each r $(0\leqq r<R)$ the integral equation

$$q(t; r)+\int_{-r}^{r}H(|t-s|)q(s; r)ds=1$$

has exactly one solution $q(t; r)$ $(0\leqq t\leqq r;\ 0\leqq r<R)$. Then this function "generates" a differential equation

$$\frac{d}{dr}\left(p\frac{dy}{dr}\right)+\kappa^2py=0\quad(0\leqq r<R),$$

where $p(r)=q^2(r; r)$ and with solutions

$$\phi(r, \kappa^2)=\frac{d}{dr}\int_0^r q(s; r)\cos\kappa s\,ds/p(r),$$

$$\psi(r, \kappa^2)=\frac{d}{dr}\int_0^r q(s; r)\omega(s, \kappa)ds/p(r),$$

where $\kappa\omega(t, \kappa)=\sin\kappa t+\int_0^t H(t-s)\sin\kappa s\,ds$ and $\phi(0; \kappa^2)=1$, $\phi'=0$, $\psi=0$, $\psi'=1$.

The case

$$q(t; r)+\int_0^r F(t, s)q(s; r)ds=1,$$

where $F(t, s)=H(|t-s|)+H(t+s)$ is also discussed. It leads to the differential equation $z''-V(r)z+\lambda z=0$.

N. Levinson (Cambridge, Mass.).

Vasilache, Sergiu. Le problème de Dirichlet pour l'équation intégro-différentielle du type elliptique. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 4, 661–668 (1952). (Romanian. Russian and French summaries)

Basing himself on a theorem established previously [Comunicările Acad. Repub. Pop. Române 1, p. 399ff. (1951); unavailable for review], the author solves Dirichlet's problem for the elliptic integro-differential equation

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+a(x, y)\frac{\partial U}{\partial x}+b(x, y)\frac{\partial U}{\partial y}+c(x, y)U$$
$$=f(x, y)+\iint_{D_P}[K(M, P)D_PU(P)]dP,$$

where

$$[K(M, P)D_P(P)]=\sum_{i,j=0}^{2}K_{ij}(x, y; \xi, \eta)\frac{\partial^{i+j}U}{\partial\xi^i\partial\eta^j}(\xi, \eta).$$

The integro-differential equation in question is first reduced to an integro-differential equation in which partial derivatives occur only under the integral sign. This second equation, which the author states may also be solved by successive approximations, is then transformed into an integral equation for which it is shown that the Fredholm theory is applicable. *J. B. Diaz* (College Park, Md.).

Vasilache, Sergiu. Sur certaines méthodes de résolution des équations intégro-différentielles, à une seule variable et à limites fixes d'intégration. Acad. Repub. Pop. Române. Stud. Cerc. Mat. 2, 287–321 (1951). (Romanian. Russian and French summaries)

The integro-differential equation considered is of Fredholm type, i.e.,

$$(1)\qquad \sum_{i=0}^{n}H_i(x)y^{(i)}(x)=f(x)+\lambda\int_a^b\sum_{r=0}^{m}K_r(x, s)y^{(r)}(s)ds.$$

[For integro-differential equations previously considered by the author see these Rev. 13, 354; 15, 630.] For the initial condition (Cauchy) problem Andreoli has reduced the problem to the solution of an ordinary Fredholm integral equation in $y^{(k)}(x)$, k the larger of m and n. The author uses a system of fundamental solutions of the differential equation $\sum_{i=0}^{n}H_i(x)y^{(i)}(x)=0$ to reduce the solution of (1) to

that of an equation in the form

$$(2)\qquad y(x)=h(x)+\lambda\int_a^b\sum_{r=0}^m P_r(x,s)y^{(r)}(s)ds,$$

where $h(x)$ depends on n arbitrary constants. This equation can be solved by successive substitutions, or equivalently by setting $y(x)=\sum_{n=0}^{\infty}\lambda^n y_n(x)$. As an alternative method applied to (2), the term $\int_a^b P_m(x,s)y^{(m)}(s)ds$ is integrated by parts, the resulting equation differentiated m times, the values of $y^{(m)}(a)$ and $y^{(m)}(b)$ determined and a new equation

$$y(x)=F(x;\lambda)+\lambda\int_a^b\sum_{r=0}^{m-1}R_r(x,s;\lambda)y^{(r)}(s)ds$$

obtained involving at most the $(m-1)$th derivative of $y(x)$; $F(x,\lambda)$ and $R(x,s,\lambda)$ rational in λ. Repeated application yields the integral equation

$$y(x)=G(x;\lambda)+\int_a^b S(x,s;\lambda)y(s)ds.$$

Another alternative method for solution of (2) involves differentiating the equation m times, solving for $y^{(m)}(x)$ in terms of the other derivatives, substituting back into (2) resulting in an equation involving at most the $(m-1)$th derivative of $y(x)$ and repeating the process.

T. H. Hildebrandt (Ann Arbor, Mich.).

Functional Analysis, Ergodic Theory

Finkbeiner, D. T., and Nikodým, O. M. On convex sets in abstract linear spaces where no topology is assumed (Hamel bodies and linear boundedness). Rend. Sem. Mat. Univ. Padova 23, 357–365 (1954).

This note continues a study initiated by Berg and Nikodým [C. R. Acad. Sci. Paris 235, 1005–1007, 1096–1097 (1952); these Rev. 14, 383]. For each Hamel basis H of a real linear space L, the associated "Hamel body" is defined to be the convex hull of $H\cup -H$. For complementary subspaces L_1 and L_2 of L and a subset S of L, S is "(L_1, L_2)-symmetric" provided $S=\{x_1-x_2: x_i\in L_i, x_1+x_2\in S\}$. The authors prove: (1) Each Hamel body is a linearly bounded closed convex body, and each convex body is the union of all translated Hamel bodies contained in it. (2) For each convex body B in an infinite-dimensional linear space L, there are a Hamel body G and a hyper-plane P such that $B+G=L=B+P$. (3) If B is a linearly bounded convex body in an infinite-dimensional linear space, there is a hyperplane P in L such that for no line λ complementary to P is it true that B is (λ, P)-symmetric.

Reviewer's comments: (i) The result (2) is obtained in a weaker form by the authors, but can be proved as stated. (ii) The paper's example of a linearly bounded convex body not contained in any Hamel body is in error, but by a different argument it can be proved that a linear space L contains such a body if and only if the Hamel dimension of L is uncountable. (The supposed example in the paper is actually a linearly bounded convex set not contained in any convex body other than the whole space.) (iii) By a more complicated argument, (3) can be strengthened to (3'): If S is an infinite-dimensional linearly bounded convex subset of a linear space L, there is a hyperplane P in L such that for no line λ complementary to P is S either (λ, P)-symmetric or (P, λ)-symmetric. (iv) It can be proved that if E is a normed linear space with unit cell C, then E is a hyper-Hilbert space if and only if for each closed hyperplane P there is a line λ complementary to P such that C is either (λ, P)-symmetric or (P, λ)-symmetric. From (3') it follows that with "closed" omitted, this condition characterizes the finite-dimensional Euclidean spaces.

V. L. Klee (Seattle, Wash.).

Andô, Tsuyoshi. Note on linear topological spaces. Proc. Japan Acad. 30, 435–436 (1954).

Let E and F be locally convex topological linear spaces, let $L(E, E)$ be the ring of continuous linear transformations from E into E, and let $L(F, F)$ be defined similarly. The author now proves two theorems, the first of which is due to Y. Kawada for normed spaces [Proc. Imp. Acad. Tokyo 19, 616–621 (1943); these Rev. 7, 306], and is shown to hold with essentially unchanged proof in general. Theorem 1. If $L(E, E)$ and $L(F, F)$ are algebraically isomorphic, then there exist algebraic isomorphisms ϕ of E onto F, and $\bar\phi$ of E' onto F', such that $\langle x, x'\rangle=\langle\phi(x), \bar\phi(x')\rangle$ for all $x\in E$ and $x'\in E'$. Theorem 2. Let $L(E, E)$ carry the topology of uniform convergence on some collection of bounded subsets of E whose union is E, and similarly for $L(F, F)$. If $L(E, E)$ and $L(F, F)$, thus topologized, are algebraically and topologically isomorphic, then so are E and F.

E. Michael.

Jerison, Meyer. A property of extreme points of compact convex sets. Proc. Amer. Math. Soc. 5, 782–783 (1954).

Let E be a locally convex, linear Hausdorff space. If $\{S_n\}$ is a sequence of sets of E, $\limsup_n S_n$ means the set of all x such that every neighborhood of x meets infinitely many of the S_n; if $S\subseteq E$, $S^\blacktriangle$ means the closed convex hull of S. Theorem: Let $\{K_n\}$ be a sequence of compact convex subsets of E whose union is contained in a convex compact subset of E. For each n let A_n be the set of extreme points of K_n, and set $K=\limsup_n K_n$, $A=\limsup_n A_n$. Then $K\subseteq A^\blacktriangle$; if also K is convex, then $K=A^\blacktriangle$; that is, A contains the set of extreme points of K.

M. M. Day.

Ishihara, Tadashige. On multiple distributions. Osaka Math. J. 6, 187–205 (1954).

This paper treats the relationship between the formally similar notions of one-parameter family of (Schwartz) distributions in n-space on the one hand and distribution in $(n+1)$-space on the other. The results are applied to show that certain equations of evolution of a type considered by Schwartz [Ann. Inst. Fourier, Grenoble 2, 19–49 (1951); these Rev. 13, 242] involving differentiation of a one-parameter family of distributions in n-space, are valid as equations for a distribution in $(n+1)$-space, and conversely under suitable additional hypotheses.

I. E. Segal.

✱Wermer, John. Invariant subspaces of bounded operators. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 314–316 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author carries further his investigations [Duke Math. J. 19, 615–622 (1952); these Rev. 14, 384] into non-trivial closed subspaces of a complex Banach space invariant under a bounded linear operator whose spectrum lies on the unit circle.

A. F. Ruston (Sheffield).

Arens, Richard, and Singer, I. M. Function values as boundary integrals. Proc. Amer. Math. Soc. 5, 735–745 (1954).

Let S be a topological space and let H be a set of bounded continuous functions on S. Šilov defined the T-frontier of

the pair S, H in case H is an algebra. Mil'man [Doklady Akad. Nauk SSSR (N.S.) **57**, 119–122 (1947); these Rev. **9**, 192] showed that if H is a suitable linear subspace of $C(S)$, then Šilov's T-frontier becomes the weak*-closure of the set of extreme maximal ideals. The present authors assume that H is a semigroup of non-negative functions, and give conditions under which an analogous compact frontier, B can be found in S itself. Under suitable assumptions it is then shown that for each s in S there is a measure m_s on B such that $\log|g(s)|=\int_B \log|g(s)|m_s(db)$ if g and g^{-1} both belong to H. Uniqueness and other properties of these measures are discussed. The results are designed to include also an inequality of Szegö [Math. Ann. **84**, 232–244 (1921)] for regular functions continuous on the disk.

M. M. Day (Urbana, Ill.).

Miyadera, Isao. On the generation of a strongly ergodic semi-group of operators. Proc. Japan Acad. **30**, 335–340 (1954).

This paper contains a summary of results of the paper reviewed below. *R. S. Phillips* (Los Angeles, Calif.).

Miyadera, Isao. On the generation of a strongly ergodic semi-group of operators. Tôhoku Math. J. (2) **6**, 38–52 (1954).

A one-parameter family of bounded linear operators $[T(\xi);\xi>0]$ on a complex Banach space $\mathfrak{X}$ to itself which satisfies the conditions: (a) $T(\xi+\eta)=T(\xi)T(\eta)$, (b) $T(\xi)$ is strongly measurable on $(0,\infty)$, (c) $\int_0^1\|T(\xi)\|\,d\xi<\infty$, is said to be a semi-group of class $(1,A)$ if

$$\text{(d)}\qquad \lim_{\lambda\to\infty}\lambda\int_0^\infty \exp(-\lambda\xi)T(\xi)x\,d\xi=x$$

for each $x\in\mathfrak{X}$, and a semi-group of class $(1,C)$ if (e) $\lim_{\eta\to 0+}\eta^{-1}\int_0^\eta T(\xi)x\,d\xi=x$ for each $x\in\mathfrak{X}$. The author obtains necessary and sufficient conditions that a linear operator A be the infinitesimal generator of a semi-group for both semi-groups of class $(1,A)$ and semi-groups of class $(1,C)$. These results are similar to those announced by the reviewer [Bull. Amer. Math. Soc. **59**, 80 (1953)] and later published [Ann. of Math. (2) **59**, 325–356 (1954); these Rev. **15**, 718]. In the case of the $(1,C)$ theorem, the reviewer's result is somewhat stronger in that it replaces Miyadera's condition (iv)—there exists a non-negative function $f(\xi,x)$ satisfying the properties: (a′) for each $x\in\mathfrak{X}$, $f(\xi,x)$ is a measurable function of ξ, (b′) $f(\xi)=\sup_x f(\xi,x)/\|x\|$ is integrable on any finite interval $[0,\epsilon]$ and bounded and measurable on any infinite interval $[\epsilon,\infty]$, $\epsilon>0$,

$$\text{(c}'\text{)}\qquad \sup[f(\xi,R(1;A)x)/\|x\|;\,x\in\Sigma].$$

is bounded on $[0,\infty)$,

$$\text{(d}'\text{)}\qquad \|R^{(k)}(\lambda;A)x\|\leqq\int_0^\infty \exp(-\lambda\xi)\,\xi^k f(\xi,x)\,d\xi$$

for all $\lambda>0$, $k\geqq 0$, and $x\in\mathfrak{X}$—by the condition: there exists a non-negative measurable function $\varphi(\xi)$ with

$$\int_0^\infty \exp(-\omega\xi)\varphi(\xi)\,d\xi<\infty$$

for some ω such that $\|R^{(k)}(\lambda;A)\|\leqq\int_0^\infty\exp(-\lambda\xi)\xi^k\varphi(\xi)\,d\xi$ for all real $\lambda>\omega$ and integers $k\geqq 0$. The corresponding theorems for semi-groups of class $(1,A)$ are somewhat different in that Miyadera deals directly with the infinitesimal generator whereas the reviewer deals with the least closed extension of the infinitesimal generator, thereby obtaining simpler sufficiency criteria. In determining sufficiency criteria for the not necessarily closed infinitesimal generator, Miyadera introduces an auxiliary subspace Σ containing $\mathfrak{D}(A)$ and which he assumes to be a Banach space under the norm

$$N(x)=\sup\left[\left\|k^{-1}\sum_{j=1}^k[\lambda R(\lambda;A)]^jx\right\|;\,k\geqq 1,\,\lambda>0\right];$$

he further assumes that $\mathfrak{D}(A)$ is dense in this space. Here again it should be noted that the author's condition (iv) can be simplified as in the $(1,C)$ theorem. *R. S. Phillips.*

Jacobs, Konrad. Ein Ergodensatz für beschränkte Gruppen im Hilbertschen Raum. Math. Ann. **128**, 340–349 (1954).

The author proves for a Hilbert space H, or for L^p, $1<p<\infty$, the following theorem: Let G be a bounded group of linear operators in H; then for each x in H there is a unique point Px, fixed under all g in G, and belonging to $K(x)$, the closed convex hull of the orbit $\{gx|g\in G\}$.

Though not so stated in the paper, the basic new idea of the proof could take the form of a Lemma: Let G be a bounded group of linear operators in a uniformly convex space B; then the space can be given an isomorphic norm invariant under every g in G and uniformly convex. Therefore, if B and B^* are isomorphic to uniformly convex spaces, the Lemma and the Alaoglu-Birkhoff ergodic theorem [Ann. of Math. (2) **41**, 293–309 (1940); these Rev. **1**, 339] show existence of the desired fixed points for both G and the adjoint group G^*; the author's device for getting uniqueness for G from existence for G^*, then works again.

M. M. Day (Urbana, Ill.).

Vaĭnberg, M. M. The variational theory of the eigenfunctions of nonlinear integral and other operators. Trudy Moskov. Mat. Obšč. **3**, 375–406 (1954). (Russian)

In several earlier papers the author used the variational method to prove the existence of characteristic elements (or invariant directions) of arbitrary norm for operators that are the gradients of functionals in $L_{2,n}$ and satisfy additional conditions [Mat. Sbornik N.S. **26(68)**, 365–394 (1950); **30(72)**, 3–10 (1952); these Rev. **12**, 340; **13**, 658]. In this paper he extends the results to $L_{p,n}$ $(p\geqq 2)$ and also generalizes his work on characteristic elements of operators in Hilbert space of the form AF, where A is self-adjoint and F is a gradient [Uspehi Matem. Nauk (N.S.) **7**, no. 2(48), 197–200 (1952); these Rev. **14**, 55], no longer requiring that A be positive nor completely continuous.

M. Golomb (Lafayette, Ind.).

Nemyckiĭ, V. V. Correction to the paper "Structure of the spectrum of nonlinear completely continuous operators". Mat. Sbornik N.S. **35(77)**, 174 (1954). (Russian)

See Mat. Sbornik N.S. **33(75)**, 545–558 (1953); these Rev. **15**, 719.

Szökefalvi-Nagy, Béla. Contractions and positive-definite operator-functions in Hilbert space. Magyar Tud. Akad. Mat. Fiz. Oszt. Közleményei **4**, 189–204 (1954). (Hungarian)

Hungarian version of Acta Sci. Math. Szeged **15**, 87–92 (1953); these Rev. **15**, 326.

Fichera, Gaetano. Formule di maggiorazione connesse ad una classe di trasformazioni lineari. Ann. Mat. Pura Appl. (4) **36**, 273–296 (1954).

For a one-to-one linear transformation T defined on a linear subspace of a Hilbert space, the author obtains

bounds for the maximum and minimum of the expression $\|u\|/\|T(u)\|$ in terms of certain related transformations. The extreme values are supposed to be assumed. Conditions ensuring this last hypothesis are quoted from earlier work. Applications are made to operators associated with elliptic and with parabolic partial differential equations.

L. M. Graves (Chicago, Ill.).

Loud, W. S. A non-exceptional element of Wiener space. Proc. Amer. Math. Soc. **5**, 940–941 (1954).

Let C denote the space of functions $x(t)$, continuous on $0 \leqq t \leqq 1$ with $x(0)=0$. P. Levy [Amer. J. Math. **62**, 487–550 (1940); these Rev. **2**, 107] has shown that for almost all elements of C, in the sense of Wiener measure, the limit $\lim_{n\to\infty} \sum_1^{2^n}[x(j/2^n)-x((j-1)/2^n)]$ exists and has the value $\frac{1}{2}$. In the paper being reviewed the author points out that for most "ordinary" functions the value of limit is zero, this being the case for example if $x(t)$ is of bounded variation, or if it satisfies a uniform Lipschitz condition of order $\alpha > \frac{1}{2}$ on $0 \leqq t \leqq 1$. The author then gives an example of an element of C in which the above limit is equal to $\frac{1}{2}$; his example is the function $x_0(t) = \frac{1}{2}\sum_1^\infty 2^{-m/2} g(t, 2^{-m})$, where $g(t, h)$ is any function which is periodic of period $2h$, equal to zero for even multiples of h, equal to one for odd multiples of h, and linear between.

W. T. Martin (Cambridge, Mass.).

Cameron, R. H. The translation pathology of Wiener space. Duke Math. J. **21**, 623–627 (1954).

Denote by m_w Wiener's measure on the space C of continuous functions $x(t)$ defined on the interval $I: 0 \leqq t \leqq 1$ and vanishing at $t=0$. It is known that measurability (but not measure) is preserved under translations $T_z: y(t) = x(t) + z(t)$ by sufficiently smooth functions $z(t)$. This set of smooth functions, however, is a set of measure zero in the space. In the paper being reviewed the author asks whether a larger class of functions gives translations which preserve measurability. He answers this question by showing that almost no translation in Wiener space preserves measurability. Having this result the author then raises the question whether there may be other measures m on C which are translation invariant. He remarks that any such measure m would have to be rather pathological and proves the following theorem: There exists no σ-finite translation invariant measure m in the space C such that every Wiener-measurable set is m-measurable and such that Wiener-measure is absolutely continuous with respect to m-measure on the σ-ring of Wiener-measurable sets. The proof is based upon a lemma which he proves on translations of the form $T_{y \tan\theta}$ applied to sets $C_{\sec\theta}$ for $0 \leqq \theta < \pi/2$, where for $0 \leqq \lambda$ the set C_λ is defined as the x's for which the limit in the preceding review exists and equals $\frac{1}{2}\lambda^2$. *W. T. Martin.*

Hewitt, Edwin, and Hirschman, Isidore, Jr. A maximum problem in harmonic analysis. Amer. J. Math. **76**, 839–852 (1954).

Let G be a locally compact abelian group, G^* its character group. Denote by f^* the Fourier transform of the (suitably restricted) function f on G. This paper deals with the cases of equality in the familiar majorisation $\|f^*\|_q \leqq \|f\|_p$, where $1<p<2$, $1/p+1/q=1$, $f \varepsilon L_p(G)$ and the norms are the usual ones on the appropriate Lebesgue spaces constructed relative to Haar measures. An $f \varepsilon L_p(G)$ for which equality holds is termed maximal in $L_p(G)$. The case in which G is the compact circle group was discussed by Hardy and Littlewood [Math. Ann. **97**, 159–209 (1926)], who showed that, for any p satisfying $1<p<2$, the maximal functions in $L_p(G)$ are precisely the scalar multiples of the characters of G. The authors extend this result to arbitrary G, for which it is necessary to introduce the so-called sub-characters: these are the scalar multiples of functions on G of the form $x \to (x, y)\varphi_A(x)$, where (x, y) is the value at x of the character $y \varepsilon G^*$, A is a compact and open subgroup of G, and φ_A is the characteristic function of A. Main theorem: For any p, $1<p<2$, the maximal functions in $L_p(G)$ are precisely the scalar multiples of subcharacters and the translates thereof. A preliminary step in the proof is the determination of cases of equality in Hölder's inequality, whilst the main stage invokes complex function theory and the maximum modulus principle in a manner reminiscent of recent proofs of the Riesz Convexity Theorem. Various examples are given: G compact; $G=R^n$; G the additive group of p-adic numbers; and G the direct product of denumerably many copies of the multiplicative group $\{-1, +1\}$. *R. E. Edwards.*

Darsow, W. F. Positive definite functions and states. Ann. of Math. (2) **60**, 447–453 (1954).

Soient G un groupe localement compact, e son élément neutre, P l'ensemble des fonctions continues de type positif sur G, L l'ensemble des fonctions continues à support compact, Q l'adhérence dans P de $L \cap P$ pour la topologie de la convergence compacte, A la C^*-algèbre engendrée par la représentation régulière gauche $f \to L_f$ de $L^1(G)$ dans $L^2(G)$, Γ l'ensemble des formes linéaires continues positives sur A. Il existe une application biunivoque "convexe-linéaire" $\varphi \to \varphi'$ de Q sur Γ telle que $\varphi'(L_f) = \langle f, \varphi\rangle$ pour $f \varepsilon L^1(G)$, et telle que $\varphi(e) = \sup_{0 \leq T \leq 1, T \varepsilon A} \varphi'(T)$. L'auteur prouve que $Q \neq P$ si G est le groupe libre à $n > 2$ générateurs d'ordre 2. [Le même résultat pour le groupe libre à 2 générateurs a été publié par H. Yoshizawa, Osaka Math. J. **3**, 55–63 (1951); ces Rev. **13**, 10.] *J. Dixmier* (Dijon).

Matsushita, Shin-ichi. Positive linear functionals on self-adjoint B-algebras. Proc. Japan Acad. **29**, 427–430 (1953).

The author states some rather complicated non-commutative generalizations of some results of Kadison [Mem. Amer. Math. Soc. no. **7** (1951); these Rev. **13**, 360], his main interest being in relations between maximal ideals in B^* algebras and extreme points of sets of functionals. He introduces the notion of "weakly extreme" functional (too complicated to be given here) which is to be a generalization of extreme functional. Although he can not prove in general that extreme implies weakly extreme, he does so claim for certain special cases. In the case for the group algebra of a locally compact group, his extreme points consist of the elementary continuous positive definite functions and the weakly extreme points are the segments combining each extreme point and 0. No proofs are given.

E. L. Griffin, Jr. (Ann Arbor, Mich.).

Wright, Fred B. A reduction for algebras of finite type. Ann. of Math. (2) **60**, 560–570 (1954).

The author provides an algebraic reduction theory for certain AW^*-algebras in the sense of Kaplansky [Ann. of Math. (2) **53**, 235–249 (1951); these Rev. **13**, 48]. Using the notion of "p-ideal" introduced by Kawada, Higuti, and Matusima [Jap. J. Math. **19**, 73–79 (1944); these Rev. **8**, 35], he first proves a non-commutative form of Stone's Theorem [Trans. Amer. Math. Soc. **41**, 375–481 (1937), p. 473] pairing the closed ideals of continuous function rings with the closed sets in the spaces on which the function rings are defined. Using the above material, he proves the follow-

ing main theorems: 1. If A is an AW^*-algebra of type II with a trace, and if M is any maximal, two-sided ideal of A, then A/M is an AW^*-factor of type II with a trace. 2. If A is a finite AW^*-algebra of type I, and if X is the set of all maximal two-sided ideals of A; then A/M is a finite AW^*-factor of type I except possibly for a nowhere dense set in X (Stone topology). This exceptional set is empty if and only if the number of homogeneous summands of A is finite. If this set is not empty, then A/M is an AW^*-factor of type II_1 for every M in this set, and A/M has a trace.
E. L. Griffin, Jr. (Ann Arbor, Mich.).

Umegaki, Hisaharu. Note on irreducible decomposition of a positive linear functional. Kōdai Math. Sem. Rep. **1954**, 25–32 (1954).

Using established techniques the author extends known results dealing with such matters as the decomposition of a group-invariant linear functional on an operator algebra into irreducible (=ergodic) parts. *I. E. Segal.*

Fukamiya, M., Misonou, Y., and Takeda, Z. On order and commutativity of B^*-algebras. Tôhoku Math. J. (2) **6**, 89–93 (1954).

Let A be a B^*-algebra with an identity and let H be the set of self-adjoint elements of A. Say that A satisfies the decomposition property if given positive a, b, c with $a \leqq b+c$ there exist positive a_1, a_2, with $a=a_1+a_2$, $a_1 \leqq b$, and $a_2 \leqq c$. Each of the authors gives a separate proof that A satisfies the decomposition property only when A is commutative. Misonou's proof uses approximations by linear combinations of projections. Fukamiya shows that each extreme state must be a homomorphism. Takeda shows that the second conjugate space of H is isomorphic as an ordered Banach space to the set of self-adjoint elements in the weak-closure of a particular operator representation of A, and concludes by citing results of Kadison [Mem. Amer. Math. Soc. no. **7** (1951), lemma 5.1; these Rev. **13**, 360] and Sherman [Amer. J. Math. **73**, 227–232 (1951), theorem 1; these Rev. **13**, 47]. *J. A. Schatz* (Bethlehem, Pa.).

Thoma, Elmar. Über vollständige Erweiterungen linearer, stetiger Abbildungen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 77–80 (1954).

Let R be a real, σ-complete, vector lattice, let $w(\mathfrak{M})$ be the space of real, finite functions on an abstract set $\mathfrak{M}$ and let $F \equiv w(\mathfrak{M})|F|r$ be a linear, continuous mapping from a sub-lattice r of R to $w(\mathfrak{M})$. Two methods are sketched of extending F to $w(\mathfrak{M})|I|s$ and $w(\mathfrak{M})|S^*|r^*$ respectively, with $r \subset s$, $r \subset r^*$; I is always an extension of S^* but it is not known whether the extension is ever a proper one. The author refers for detailed proofs to his dissertation [Erlangen, 1952]. He also refers to H. Nakano [Proc. Imp. Acad. Tokyo **19**, 138–143 (1943); these Rev. **8**, 387] although he became aware of Nakano's work only after his own paper was in proof. *I. Halperin* (Utrecht).

Theory of Probability

Popper, Karl R. Degree of confirmation. British J. Philos. Sci. **5**, 143–149 (1954).

The author argues that the intuitive idea of the degree of confirmation of a hypothesis x by evidence y refers to the extent to which x is supported by y, and should not be defined as the conditional probability $P(x|y)$, as is done by R. Carnap. The author gives several desiderata that he believes should be satisfied by degree of confirmation, one of them being that the maximum degree of confirmation that x can have should increase with $P(\bar{x})$, where the bar represents negation (*). He then suggests, as one possible definition,

$$[1+P(x)P(x|y)][P(y|x)-P(y)]/[P(y|x)+P(y)].$$

In the reviewer's opinion the simpler expression

$$\log [P(x|y)/P(x)],$$

the "amount of information in y concerning x", would satisfy an even more convincing set of desiderata, and the maximum degree of confirmation that x could have would be what is sometimes meant by the "amount of information" in x. Were it not for the condition (*) a satisfactory definition would be the "weight of evidence", "support", "log-factor", or logarithm of the likelihood ratio,

$$\log [P(y|x)/P(y|\bar{x})].$$

[Cf. N. Wiener, Cybernetics, Wiley, New York, 1948, p. 75; these Rev. **9**, 598; I. J. Good, Probability and the weighing of evidence, Griffin, London, 1950, pp. 62, 75; these Rev. **12**, 837; P. M. Woodward, Probability and information theory, McGraw-Hill, New York, 1953, pp. 51, 53; these Rev. **15**, 450; H. Jeffreys, Proc. Cambridge Philos. Soc. **32**, 416–445 (1936).] *I. J. Good* (Cheltenham).

✶Copeland, Arthur H., Sr. A finite frequency theory of probability. Studies in mathematics and mechanics presented to Richard von Mises, pp. 278–284. Academic Press Inc., New York, 1954. $9.00.

Observing that the ordinary frequency theory of physical probabilities (chances) makes use of infinite sequences of trials which cannot occur in practice, the author suggests what he calls a "finite frequency theory". Probabilities are to be regarded as mental attitudes that are estimates of probabilities occurring in statistical hypotheses. These estimates can be tested by the ordinary method of statistics, i.e. by means of finite sequences of trials, and rejected at various conventional significance levels.

More generally the theory contains a recommendation to say that a proposition x is true if its estimated probability exceeds λ, where rules for the selection of λ are not given, but there is a suggestion that $\lambda=0.95$ is sometimes a reasonable value to use. We may say that x is true and y is true without implying that y is true.

The formal apparatus is expressed in terms of a Boolean algebra of propositions as in an earlier paper of the author [Math. Z. **53**, 285–290 (1950); these Rev. **12**, 721].
I. J. Good (Cheltenham).

Oswald, F., and Schuh, Fred. A problem from the calculus of probabilities. Simon Stevin **30**, 106–112 (1954). (Dutch)

A sample of size n is drawn from an infinite discrete rectangular universe with k equidistant variate values and mean m. Let X be the number of sample values equal to an assigned number a, the remainder being supposed to exceed a. Consideration is given to the behavior of the expected value of $(n+1)/(x+1)$ when $k=6$, a is near m and $n=2, 3, \cdots$.
H. L. Seal (New York, N. Y.).

Holte, Fritz C. Some properties of the binomial distribution. Nordisk Mat. Tidskr. **2**, 113–115, 136 (1954). (Norwegian. English summary)

Generalizes, for a short range of values of p, a result of Frisch [Skand. Aktuarietidskr. **7**, 153–174 (1924)] on the

comparative sizes of the sum of the binomial terms preceding the mode ($p<\frac{1}{2}$) and the sum of the terms following it.

H. L. Seal (New York, N. Y.).

★Garti, Y., et Consoli, T. Sur la densité de probabilité du produit de variables aléatoires de Pearson du type III. Studies in mathematics and mechanics presented to Richard von Mises, pp. 301–309. Academic Press Inc., New York, 1954. $9.00.

Let x, y, z be independently distributed chance variables having Gamma distributions. The distribution, moments, and characteristic function of $u=xy$ are derived. An approximation to the distribution of $v=xyz$ is given.

G. E. Noether (Boston, Mass.).

★Frenkiel, François N., and Follin, James W., Jr. On multivariate normal probability distributions. Studies in mathematics and mechanics presented to Richard von Mises, pp. 295–300. Academic Press Inc., New York, 1954. $9.00.

Let x, y, z be three independent random variables normally distributed and having mean values μ_x, μ_y, μ_z and standard deviations σ_x, σ_y, σ_z, respectively. The authors determine: (a) the probability density function of the random variable $(x^2+y^2)^{1/2}$ when $\mu_x\neq 0$, $\mu_y=0$ and $\sigma_x\neq\sigma_y$; (b) the probability density function of the random variable $(x^2+y^2+z^2)^{1/2}$ when $\mu_x\neq 0$, $\mu_y=\mu_z=0$ and $\sigma_x\neq\sigma_y=\sigma_z$.

M. Muller (Ithaca, N. Y.).

Plackett, R. L. A reduction formula for normal multivariate integrals. Biometrika 41, 351–360 (1954).

Let $(X_1, X_2, \cdots, X_n)$ be a vector of chance variables with a nonsingular multivariate normal distribution. The problem is to evaluate $P(X_1>a_1, \cdots, X_n>a_n)$. The author obtains a reduction formula for this probability, involving integrals of partial derivatives of the probability with respect to the elements of the covariance matrix of $(X_1, \cdots, X_n)$. For $n=3$ and $n=4$, the reduction formula enables the author to express the probability as a finite sum of single integrals of tabulated functions. These integrals have to be evaluated by numerical quadrature, but for certain cases simple approximations to them are given.

L. Weiss (Charlottesville, Va.).

Lord, R. D. The distribution of distance in a hypersphere. Ann. Math. Statistics 25, 794–798 (1954).

Let r denote the distance between two points A and B that are independently and uniformly distributed in a hypersphere in s dimensions. For the distribution of r J. M. Hammersley [same Ann. 21, 447–452 (1950); these Rev. 12, 268] has obtained, using special methods, a formula expressed in terms of the incomplete Beta function. The author obtains this formula by two different methods: (i) by use of characteristic functions in polar form and of Hankel transforms; (ii) by projection from a space of $s+2$ dimensions onto a subspace of s dimensions. Both his methods are applicable also in the more general case where the distribution of A (or B) is spherical, that is, where its density depends only on distance from the origin. Method (ii) connects the problem with a known result on random flights [cf. G. N. Watson, A treatise on the theory of Bessel functions, Cambridge, 2nd ed., 1944, §§13.48, 13.46 (3); these Rev. 6, 64].

H. P. Mulholland (Birmingham).

Krishna Iyer, P. V., and Prakasa Rao, A. S. Theory of the probability distribution of runs in a sequence of observations. J. Indian Soc. Agric. Statistics 5, 29–77 (1953).

The first two moments of and the covariances between the number of junctions and ascending, descending, and stationary runs are obtained (1) when $X_1, \cdots, X_n$ are independently and identically distributed and can take on only k different values with given probabilities, and (2) in the conditional population where the number of X_i taking on each of the k values is given. Recurrence relations between generating functions of probability distributions are given, and the distributions are given for small n. Statistical uses (e.g., to test randomness) are suggested. In most previous work on this subject it has been assumed that there are no stationary runs, i.e., that the X_i have a continuous distribution.

J. Kiefer (Ithaca, N. Y.).

Lukacs, Eugène. Sur une caractérisation de la distribution de Poisson. C. R. Acad. Sci. Paris 239, 1114–1116 (1954).

Let $X_1, \cdots, X_n$ be independent non-negative random variables with the same distribution, whose third moment is to exist. If $x_1, \cdots, x_n$ are observations on $X_1, \cdots, X_n$, let $L=nk_1=x_1+\cdots+x_n$ and $S=k_3-k_1$, where k_r denotes the usual unbiased estimate for the rth cumulant. The author proves that a necessary and sufficient condition for the X's to have a Poisson distribution is that the regression of S upon L should be constant, that is, that the conditional expectation $E(S|L)$ should equal $E(S)$. [For a characterization of the normal distribution on similar lines cf. R. G. Laha, Biometrika 40, 228–229 (1953); these Rev. 15, 725.] The proof depends on the observation that $E(S|L)=E(S)$ if and only if $E(Se^{itL})=E(S)E(e^{itL})$ $(-\infty<t<\infty)$, and upon a lemma due to R. P. Boas [C. R. Acad. Sci. Paris 228, 1837–1838 (1949), p. 1837; these Rev. 11, 27]. [Remark: in the author's arguments we can replace S by $T=k_2-k_1$ if we suppose now that the parent distribution lies in $[a, \infty)$ when $a=0$ but not when $a>0$ and that its second moment exists.]

H. P. Mulholland.

Teicher, Henry. On the factorization of distributions. Ann. Math. Statistics 25, 769–774 (1954).

A family S of c.d.f.'s is said to be factor-closed (f.c.) if, for any element F of S, the relationship $F=G*H$ implies that G and H are members of S. The classical binomial family and certain generalizations of it are shown to be f.c. The multinomial family is also f.c. Most families of infinitely divisible distributions are not f.c.

G. E. Noether.

Teicher, Henry. On the convolution of distributions. Ann. Math. Statistics 25, 775–778 (1954).

A systematic approach to distributions having the reproductive property is attempted, and necessary and sufficient conditions are given. The case of distributions depending on more than one parameter need not be a straightforward generalization of the one-parameter case. (From the author's summary.)

G. E. Noether (Boston, Mass.).

Bergström, Harald. On some expansions of stable distribution functions. Ark. Mat. 2, 375–378 (1952).

Take $\alpha\in(0, 2)$, $\cos\beta>0$, and let $G'_{\alpha\beta}$ be the probability density of the stable distribution whose characteristic function is $\exp(-|t|^\alpha(\cos\beta-i\sin\beta\operatorname{sign}(t)))$. Generalizing a result of P. Humbert and H. Pollard [Pollard, Bull. Amer. Math. Soc. 52, 908–910 (1946); these Rev. 8, 269], the author shows that if $|x|>0$, $xG'_{\alpha\beta}(x)$ can be expanded in

powers of $|x|^{-\alpha}$, the expansion being convergent when $0<\alpha<1$ and asymptotic ($|x|\to+\infty$) when $1\leqq\alpha<2$; and that $G'_{\alpha\beta}(x)$ can be expanded in powers of x, the expansion being convergent when $1<\alpha<2$ and asymptotic ($|x|\to0$) when $0<\alpha\leqq1$.

There is a minor mistake in line 2 on page 378: there, the max should be taken over (z: Re $(z)\leqq0$), not over the whole plane. *H. P. McKean Jr.* (Princeton, N. J.).

Bergström, Harald. Eine Theorie der stabilen Verteilungsfunktionen. Arch. Math. **4**, 380–391 (1953).

The fundamental properties of stable distributions are established by partly new methods, based on the author's previous results [Skand. Aktuarietidskr. **27**, 139–153 (1944); **34**, 1–34 (1951); Ark. Mat. **2**, 463–474 (1953); these Rev. **7**, 458; **13**, 258; **15**, 237; see also the paper reviewed above]. *K. L. Chung* (Syracuse, N. Y.).

Linnik, Yu. V. On stable probability laws with exponent less than one. Doklady Akad. Nauk SSSR (N.S.) **94**, 619–621 (1954). (Russian)

Three theorems are proved about the asymptotic behavior near 0 of the density function of stable distributions with exponent $a<1$ which vanishes for $x<0$. The first gives the principal term with an error estimate. The second seems to be a less precise form of the Humbert-Pollard expansion [Pollard, Bull. Amer. Math. Soc. **52**, 908–910 (1946); these Rev. **8**, 269]. Neither these authors nor the recent work of Bergström [see the second preceding review] is mentioned. The third theorem establishes a rather complicated differential equation in the case a is the reciprocal of an integer. *K. L. Chung* (Syracuse, N. Y.).

✱Sparre Andersen, Erik. Some theorems on sums of symmetrically dependent random variables. Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 291–296 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

A summary of results by the author and others concerning the number of positive summands and related things, mostly already published [Skand. Aktuarietidskr. **32**, 27–36 (1949); **36**, 123–138 (1953); Math. Scand. **1**, 263–285 (1953); these Rev. **11**, 256; **15**, 634, 444]. *K. L. Chung.*

Kolmogorov, A. N. Some work of recent years in the field of limit theorems in the theory of probability. Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) **8**, no. 3(10), 5–14 (1954). (Romanian)

Translation of an expository paper in Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **8**, no. 10, 29–38 (1953).

Petrov, V. V. Generalization of Cramér's limit theorem. Uspehi Matem. Nauk (N.S.) **9**, no. 4(62), 195–202 (1954). (Russian)

The principal result is the following: Let Z_i, $i=1, 2, \cdots$, be independent chance variables, with $EZ_i=0$. Let

$$B_n=\sum_{i=1}^n EZ_i^2,$$

and $F_n(x)=P\{\sum_{i=1}^n Z_i<x\sqrt{B_n}\}$. Suppose 1) there exist positive A, K, and k such that, in the circle $|h|<A$, for $i=1, 2, \cdots, k\leqq|E\{\exp hZ_i\}|\leqq K$. 2) For every n

$$n^{-1}\sum_{i=1}^n EZ_i^2\geqq\delta>0.$$

3) For arbitrary $\epsilon>0$ there exists a $C>1$ such that for all n

$$n^{-1}\sum_{i=1}^n\int_{|y|>C}y^2dV_i(y)<\epsilon,$$

where $V_i(y)$ is the distribution function of Z_i.

Let x be a real number which depends on n, is >1 and $=o(\sqrt{n})$ as $n\to\infty$. Then

$$(1-F_n(x))\left(\frac{1}{\sqrt{(2\pi)}}\int_x^\infty e^{-\frac12y^2}dy\right)^{-1}=\exp\left\{\frac{x^3}{\sqrt{n}}\lambda_n\left(\frac{x}{\sqrt{n}}\right)\right\}\left[1+O\left(\frac{x}{\sqrt{n}}\right)\right],$$

$$(F_n(-x))\left(\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{-x} e^{-\frac12y^2}dy\right)^{-1}=\exp\left\{-\frac{x^3}{\sqrt{n}}\lambda_n\left(-\frac{x}{\sqrt{n}}\right)\right\}\left[1+O\left(\frac{x}{\sqrt{n}}\right)\right],$$

where $\lambda_n(t)$ is a power series which converges for sufficiently small $|t|$ uniformly for all n.

The theorem of Cramér [Actualités Sci. Ind., no. 736, Hermann, Paris, 1938, pp. 5–23], referred to in the title, is slightly weaker than the theorem cited above applied to identically distributed chance variables. *J. Wolfowitz.*

Steinhaus, H. Sur les fonctions indépendantes. X. Equipartition de molécules dans un récipient cubique. Studia Math. **13**, 1–17 (1953).

[For part IX see Studia Math. **12**, 102–107 (1951); these Rev. **13**, 958.] Let n particles be enclosed in a unit cube and let the velocity components of the kth particle (in directions parallel to the edges of the cube) be λ_k, μ_k, ν_k. The particles are allowed to reflect from the walls but not to collide with each other. If the $3n$ numbers λ_k, μ_k, ν_k are linearly independent it is shown that the relative time during which k particles are in a volume v is given by the binomial formula $\binom{n}{k}v^k(1-v)^{n-k}$. Remarks concerning the relation of this result to those of Egerváry and Turán [Studia Math. **12**, 170–180 (1951); these Rev. **13**, 761] are appended. *M. Kac* (Ithaca, N. Y.).

Itô, Kiyosi. Stationary random distributions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 209–223 (1954).

The author applies the Schwartz distribution theory to the study of second-order processes. In the usual approach, such a process is a Hilbert-space-element-valued continuous function $X(\cdot)$ on $(-\infty, \infty)$. If ϕ is a function in class C_∞ on $(-\infty, \infty)$, with compact carrier, $\int_{-\infty}^\infty X(t)\phi(t)dt$ defines a Hilbert-space element for each ϕ, so that the integral defines a Hilbert-space-valued continuous linear functional on the ϕ class. More generally, any such functional is called a random distribution. Stationarity, normality and other concepts useful in discussions of second-order processes are defined in a natural way. Derivatives of random distributions always exist and are new random distributions. The covariance of a stationary random distribution becomes an ordinary Schwartzian distribution on the ϕ class, and the general form of every covariance is found, involving a measure, the "spectral measure", in a generalization of the classical Khintchine formula for a covariance function. The corresponding spectral form of the stationary random distribution is also found. Finally, this work is applied to stationary random distributions and their derivatives, to obtain the general form of processes with stationary kth-

order increments in an elegant way. The case $k=1$ was treated by Kolmogorov [C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 6–9 (1940); these Rev. 2, 220]. *J. L. Doob.*

Bochner, S. Limit theorems for homogeneous stochastic processes. Proc. Nat. Acad. Sci. U. S. A. 40, 699–703 (1954).

The analytic content of the paper is as follows. Let $\phi(\cdot)$ be a k-dimensional infinitely divisible characteristic function and let $F(\cdot, t)$ be the distribution function corresponding to $[\phi(\cdot)]^t$, $G_n(\cdot)=nF(\cdot, 1/n)$. Let $\mu(\cdot)$ be a continuous function in E_k, 0 at the origin. Nine theorems are stated concerning the limiting behavior of $\int_{E_k}(1-e^{-2\pi i(\beta,\mu(x))})dG_n(x)$ as $n\to\infty$. For example, if $\mu(x)$ is linear $+o(|x|)$ near the origin, we have the canonical form of an infinitely divisible law; if $\mu(x)$ is quadratic $+o(|x|^2)$ near the origin, then the limit exists as such and we have a generalization of a theorem of Kunisawa and Maruyama [Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 1, no. 3, 22–27 (1951); these Rev. 14, 294]. Other cases are examined if F corresponds to a symmetric stable law and $\mu(x)=|x|^q$. If $x(\Delta)$ is the homogeneous process corresponding to F, then $\mu(x(\Delta))$ is a process which may be interpreted as a variation of the path of the $x(\Delta)$ process. It is not clear whether the two random variables to be added in the fourth line of the paper are supposed to be independent or not. *K. L. Chung.*

van Dantzig, D., and Scheffer, C. On arbitrary hereditary time-discrete stochastic processes, considered as stationary Markov chains, and the corresponding general form of Wald's fundamental identity. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 377–388 (1954).

Let E_i, $i=0, 1, \cdots$, be sets. E_0 consists of a single element π_0 while E_i for $i>0$ is an arbitrary set with a σ-field of subsets. A stochastic process $x_1, x_2, \cdots$, with $x_i \in E_i$, is defined by a function $P\{x_{n+1} \in X|x_1, \cdots, x_n\}$, with suitable measurability conditions. Let $\pi=(x_1, \cdots, x_n)$ represent a "path." A space F_ω of paths π is defined and the original transition function P becomes a stationary Markov transition function, in the space of paths π, satisfying the Chapman-Kolmogorov equations. A generalized form of Wald's fundamental identity is given for general processes of the above type. A concise statement of the result is difficult to give, but it bears some relation to martingale theory and includes various versions of Wald's fundamental identity as well as the converse proposition. *T. E. Harris.*

Silvey, Samuel D. A problem associated with a particular Markov chain. Proc. Glasgow Math. Assoc. 2, 100–104 (1954).

The following Markov chain is considered. The $2r+1$ states are $-r, -(r-1), \cdots, r-1, r$. The transition probabilities are

$$p(j, j+1)=p(j, j-1)=\tfrac{1}{4}, \quad p(j, j)=\tfrac{1}{2} \quad \text{for} \quad -r<j<r,$$

$p(-r, -r)=p(r, r)=\frac{3}{4}$ and $p(-r, -(r-1))=p(r, r-1)=\frac{1}{4}$. The probability associated with the ith state after t time intervals is denoted by p_{it}. Let $S_t=\sum_{i=-r}^{r} ip_{it}$. Then, for any choice of initial probabilities, S_t approaches 0 and the value of t such that $S_t=\frac{1}{2}S_0$ is called the mean half-life of the system. The author shows how to determine an approximate value for the mean half-life in the case $p_{r0}=1$ and large r. The problem is said to arise in investigations with computing machines. *J. L. Snell.*

Eberl, W. Ein Zufallsweg in einer Markoffschen Kette von Alternativen. Monatsh. Math. 58, 137–142 (1954).

Ein Teilchen, das sich zu Beginn seiner Bewegung auf der Zahlengeraden im ganzzahligen Punkt z befindet $(0<z<k)$, legt bei jedem Schritt den Weg $+1$ oder -1 mit den bezüglichen Wahrscheinlichkeiten p_1, $q_1=1-p_1$ oder q_2, $p_2=1-q_2$ zurück, je nachdem der unmittelbar vorangehende Schritt $+1$ oder -1 war. Die Punkte 0 und k sind Absorptionsschranken. Verf. ermittelt die bezüglichen Absorptionswahrscheinlichkeiten und die wahrscheinliche Dauer der Wanderung durch Lösung von einem System von Differenzengleichungen. *J. Wolfowitz* (Ithaca, N. Y.).

Takács, Lajos. A new method for discussing recurrent stochastic processes. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 135–151 (1954). (Hungarian. Russian and English summaries)

The author derives the distribution of recurrence times for two special types of recurrent events from the function determining the expected number of events during the time interval $(0, t)$. *E. Lukacs* (Washington, D. C.).

Takács, Lajos. Coincidence problems arising in the theory of counters. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 2 (1953), 153–163 (1954). (Hungarian. Russian and English summaries)

The author deals with counter problems of the type discussed in his earlier papers [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 1, 371–386 (1951); Acta Math. Acad. Sci. Hungar. 2, 275–298 (1951); these Rev. 13, 956; 14, 388] and investigates the following physical situation. Particles arrive at a counter and are observed by a registering device. It is assumed that the arrival of particles occurs according to a Poisson process and that each arrival produces at the counter a voltage pulse of random intensity. The pulses decay exponentially and are additive. The counter registers only if the voltage exceeds a certain threshold value. The author treats the problem of determining the frequency of arrivals from the frequency of the registrations and uses in his discussion some unpublished results. He also considers the case of random dead times which occur if Geiger-Müller counters are employed. Finally coincidence problems are considered. These arise if several counters are used simultaneously. *E. Lukacs* (Washington, D. C.).

Morse, Philip M., Garber, H. N., and Ernst, M. L. A family of queuing problems. J. Operations Res. Soc. Amer. 2, 444–445 (1954).

The authors consider queues with multiple service channels, a in number. The service times for each channel are assumed to be distributed as convolutions of k exponential distributions. The arrivals are in accordance with a Poisson law. Recursion formulas are given for the probabilities of states of the system. It is stated that a computation program is under way. *T. E. Harris.*

Mayne, Alan J. Some further results in the theory of pedestrians and road traffic. Biometrika 41, 375–389 (1954).

The work of Tanner [Biometrika 38, 383–392 (1951); these Rev. 13, 666] on the delay to pedestrians crossing a road carrying a random stream of cars is extended to allow car inter-arrival times with any given distribution (these times are independent and each has the same distribution). A crossing is effected only if the lane is clear for a time interval at least I; hence the pedestrian's delay is analogous

to the time of occupation of a Type II counter with dead-time I, a similarity of which the author seems incompletely aware. A solution for the Type II counter, with arbitrary inter-arrival time and arbitrary dead-time distributions, has been given by Pollaczek [C. R. Acad. Sci. Paris **238**, 766–768 (1954); these Rev. **15**, 542]. However, the Laplace transform of the delay distribution, the main result of this paper, has not been given explicitly by Pollaczek; and the further results on the sizes of pedestrian queues are of course not considered in the counter problem. In the last, the results of Tanner are again generalized by allowing pedestrian inter-arrival times with any given distribution, and the presence of islands between lanes of traffic is considered, with indications that substantial reductions of pedestrian delay are possible in a multi-lane situation.

Finally some traffic data are examined with particular attention to tests for randomness. New tests are described which are based on dividing the interval of observation into sub-intervals corresponding to equal probability intervals on the scale of the inter-arrival time distribution.

J. Riordan (New York, N. Y.).

Rényi, Alfréd. Betrachtung chemischer Reaktionen mit Hilfe der Theorie der stochastischen Prozesse. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 83–101 (1954). (Hungarian. Russian and German summaries)

The author considers the evolution of a chemical reaction as a stochastic process depending on a continuous time parameter. The merits of these methods are discussed and a number of simple probabilistic models for chemical processes is constructed. As an example we mention, that the law of mass action is derived for bimolecular reactions.

E. Lukacs (Washington, D. C.).

Prékopa, András. Statistical treatment of the degradation process of long chain polymers. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 103–123 (1954). (Hungarian. Russian and English summaries)

The author gives a stochastic model for the degradation of long chain polymers. By suitable specialization of his assumptions he obtains a system of differential equations for the expected number $N(k, t)$ of chains of length k present at time t. He remarks that these equations, as well as some other results, were already earlier obtained without a formal probabilistic justification by Simha [J. Appl. Phys. **12**, 569–578 (1941)] and by Montroll and Simha [J. Chem. Phys. **8**, 721–727 (1940)]. *E. Lukacs.*

Ammeter, H. La théorie collective du risque et l'assurance de choses. Mitt. Verein. Schweiz. Versich.-Math. **54**, 185–204 (1954).

Expository paper. *H. L. Seal* (New York, N. Y.).

Chernoff, Herman, and Lieberman, Gerald J. Use of normal probability paper. J. Amer. Statist. Assoc. **49**, 778–785 (1954).

Mathematical Statistics

Hemelrijk, J., and van Elteren, Ph. A course in applied statistics. Math. Centrum Amsterdam. Statist. Afdeling. Rep. S 120, 78 pp. (1954). (Dutch)

This is a brief course, of 78 pages and 8 sections, in statistical notions and methods, starting from the gathering of data, and concluding with an exposition of Wilcoxon's test of individual comparison by ranking methods. The other tests considered in detail are the Sign Test, illustrated by sequences of heads and tails for a tossed coin, and the Binomial Test. Formulas, tables, examples, and cautionary remarks are scatttered throughout. The last two pages consist of errata. *A. A. Bennett* (Providence, R. I.).

Epstein, Benjamin. Tables for the distribution of the number of exceedances. Ann. Math. Statistics **25**, 762–768 (1954).

Let O_{n1} and O_{n2} be two independent random samples of size n from a continuous distribution, and let U_r^n be the number of values of O_{n2} which exceed the rth smallest value of O_{n1}. The author gives a table of $\Pr(U_r^n \leq x)$ for $r=1(1)[n/2]$, $x=(r-1)(1)(n-r-1)$, and $n=2(1)15(5)20$. He further considers a two-sided exceedance problem, where O_{n1} now stands for the sample with the largest rth order statistic (counted from below). He points out the close connection between exceedance problems and problems of slippage [cf., e.g., Mosteller and Tukey, same Ann. **21**, 120–123 (1950); these Rev. **11**, 608]. The distribution of U_r^n has been studied by Gumbel and von Schelling [ibid. **21**, 247–262 (1950); these Rev. **11**, 732].

D. M. Sandelius (Göteborg).

Sarkadi, Károly. Choice of intervals for grouping of data. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 299–306 (1 plate) (1954). (Hungarian. Russian and English summaries)

The author discusses a method of grouping data for the purpose of computing the mean of a sample. The method minimizes the difference between the mean as calculated from the grouped and from the ungrouped data.

E. Lukacs (Washington, D. C.).

Rao, C. Radhakrishna. A general theory of discrimination when the information about alternative population distributions is based on samples. Ann. Math. Statistics **25**, 651–670 (1954).

Let $p_i(x|\theta_i)$ $(i=1, 2, \cdots, k)$ be probability densities with known functional forms (x and θ_i may be vector-valued) and suppose samples x_j^i $(j=1, 2, \cdots, n_i)$ are available from each of the k populations with these densities. Further, an individual known a priori to belong to one of these populations has measurements x. The problem is to assign this individual to its proper group on the basis of the observations x, x_j^i, so as to minimize some function of the errors of misclassification. The solutions of this problem for θ_i known and for θ_i unknown but n_i "large" are reviewed, and some possible small sample discrimination rules are discussed rather generally. A number of possible requirements on the decision rule are stated. These requirements, which primarily involve the local behaviour of the error functions for $\theta_i-\theta_j$ near zero, lead to various explicit equations for the decision rules, provided complete sufficient statistics are assumed to exist for the θ_i. Finally, some of these results are applied to the case that the observations are multinormal with known or unknown dispersion matrices. As the author points out, the distribution theory of the discrimination criteria is unsolved, so that the performance characteristics of these rules are completely unknown.

D. G. Chapman (Oxford).

Hayashi, Chikio. Multidimensional quantification. I. Proc. Japan Acad. **30**, 61–65 (1954).

Each element of a given set has been assigned to one stratum of a given set of strata. The problem of this paper is to obtain an efficient numerical characterization of this stratification by which a similar stratification will be assigned automatically to a similar set of elements. It is assumed that the stratification was decided on the basis of a classification of the elements with respect to various items. That is, associated with each item is a set of categories and each element belongs to just one of these categories. To each category is assigned a numerical label. Each element is assigned a score which is the sum of all the labels of those categories to which the element belongs. Each stratum is then assigned the label which is the average of the scores of all the elements of that stratum. A similar stratification of similar elements can be performed by comparing the element scores with the stratum labels. To select the category labels one forms the ratio of the variance of the stratum labels to the variance of the category labels and then maximizes this ratio with respect to the category labels. The maximum is subject to certain linear side conditions. *A. H. Copeland Sr.* (Ann Arbor, Mich.).

Hayashi, Chikio. Multidimensional quantification. II. Proc. Japan Acad. **30**, 165–169 (1954).

The purpose of this paper is similar to that of the preceding. The difference is that each stratum is characterized by a vector instead of a number. Each component of such a vector corresponds to an item. Thus if there are R items then each stratum is characterized by an R-dimensional vector. A given component of such a vector varies from one stratum to another and can be regarded as a random variable. Hence there are R random variables and an associated covariance matrix. Corresponding to the category labels is another set of R random variables and another covariance matrix. The ratio of the determinants of these covariance matrices is used in the maximization procedure instead of the ratio of the variances. The maximum together with certain linear side conditions determines the category labels. The author also considers a mixed problem in which the characterization of the stratification is intermediate between the vector characterization just described and the scalar characterization of the preceding paper.
A. H. Copeland Sr. (Ann Arbor, Mich.).

***Gini, Corrado. Estensioni e portata della teoria della dispersione.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 323–335. Academic Press Inc., New York, 1954. $9.00.

This is a discussion of dispersion from the viewpoint of descriptive statistics which utilizes various urn schemes.
E. Lukacs (Washington, D. C.).

Kamat, A. R. Moments of the mean deviation. Biometrika **41**, 541–542 (1954).

Wishart, John. The factorial moments of the distribution of joins between line segments. Biometrika **41**, 555–556 (1954).

Zia ud-Din, M. Expression of the k-statistics k_9 and k_{10} in terms of power sums and sample moments. Ann. Math. Statistics **25**, 800–803 (1954).

Explicit expressions are given for the ninth and tenth k-statistics in terms of power sums and n, the sample size.
S. W. Nash (Vancouver, B. C.).

Nelder, J. A. The interpretation of negative components of variance. Biometrika **41**, 544–548 (1954).

Horák, Zdeněk. A generalization of the normal error law. Čehoslovack. Fiz. Ž. **4**, 187–203 (1954). (English. Russian summary)

Generalizing the normal probability density, the author considers a density given by

$$\eta(x|\alpha,\beta)=a(\alpha,\beta)/(\exp\{\alpha^2x^2\}-\beta),$$

where α and $\beta<1$ are adjustable parameters. The proposed method of fitting this formula to empirical distributions is based on the average deviation and on the mean square. A numerical table is provided to facilitate the process of fitting. Several series of measurements are investigated to find out whether they can be satisfactorily fitted by a normal curve and by the author's generalization and the estimated values of β vary from -5 to $+0.71$. By visual inspection, the fit provided by the formula indicated is, in some cases, substantially closer than the fit by the normal law. *J. Neyman* (Berkeley, Calif.).

***Fréchet, Maurice. Interdépendance du centre et du rayon empiriques de variation de n observations indépendantes. I.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 285–294. Academic Press Inc., New York, 1954. $9.00.

The author proves that there exists no distribution function $F(x)$, with a bounded first derivative and a second derivative continuous everywhere, such that, for at least two integers n, $(M_n''-M_n')$ and $(M_n''+M_n')$ are independently distributed, where M_n'' and M_n' are, respectively, the largest and smallest of n independent chance variables with the common distribution $F(x)$. More general results, by weakening the restrictions on $F(x)$, are promised in a future publication. *J. Wolfowitz* (Ithaca, N. Y.).

Hill, I. D. The distribution of the regression coefficient in samples from a non-normal population. Biometrika **41**, 548–552 (1954).

The author finds the large-sample distribution of the regression coefficient $b=k_{11}/k_{20}$ in a sample from a non-normal population by fitting various standard distributions to the variate $\psi=k_{11}-ck_{20}$. Numerical comparisons give confidence in the method. *P. Whittle* (Wellington).

Chernoff, Herman. On the distribution of the likelihood ratio. Ann. Math. Statistics **25**, 573–578 (1954).

Consider testing whether parameter vector $\theta\in\omega$ against the alternative $\theta\in\tau$, where ω and τ are disjoint subsets of parameter space which can be approximated by positively homogeneous sets (i.e. by bundles of half lines starting at the origin). Assume that the density of x satisfies the usual regularity conditions. Let $L(X,\theta)$ denote the likelihood function and

$$\lambda^*(X)=\{\sup_{\theta\in\omega}L(X,\theta)\}/\{\sup_{\theta\in\tau}L(X,\theta)\}.$$

The author first discusses the distribution of $-2\log\lambda^*(X)$ for X a single normally distributed observation. Then he shows that, if $\theta=0$ is a boundary point of ω and the maximum likelihood estimator $\hat\theta_\omega\xrightarrow{P}0$ when $\theta=0$, then, when $\theta=0$, the asymptotic distribution of λ^* is the same as it would be for the test based on one observation from a normally distributed population with mean θ and covariance matrix that of $n^{1/2}\hat\theta$. *S. W. Nash* (Vancouver, B. C.).

David, F. N., and Johnson, N. L. Statistical treatment of censored data. I. Fundamental formulae. Biometrika 41, 228–240 (1954).

Let $x_1, x_2, \cdots, x_n$ be an increasingly ordered sample of a random variable x which has a probability density. Using a formal expansion in an inverse Taylor series, the authors obtain a procedure for computing approximations to the moments of the x_r's, and apply the so obtained expressions to special cases such as the median, the quartiles and their distance, the correlation between x_r and x_s. Several criteria are then considered for testing certain simple hypotheses when the available data form a censored sample such that the total sample size and the number of omitted values are fixed in advance (e.g. in life-testing if survival times of the first k individuals to die out of N are observed). Continuation of the work initiated in the present paper is announced and a number of problems are listed for further discussion.

Z. W. Birnbaum (Seattle, Wash.).

Higuchi, Isao. On the solutions of certain simultaneous equations in the theory of systematic statistics. Ann. Inst. Statist. Math. Tokyo 5, 77–90 (1954).

The author proves the existence of the optimum spacing in the use of systematic statistics treated by Ogawa [see Osaka Math. J. 3, 175–213 (1951); these Rev. 13, 762] in the univariate normal cases where either (a) the mean or (b) the variance is the unknown parameter. *R. P. Peterson.*

Lord, R. D. The use of the Hankel transform in statistics. II. Methods of computation. Biometrika 41, 344–350 (1954).

The author gives approximate methods for evaluating an inversion formula which occurs in his paper in Biometrika 41, 44–55 (1954) [these Rev. 15, 885]. Various series are used. *J. Wolfowitz* (Ithaca, N. Y.).

Korolyuk, V. S. Asymptotic expansions for A. N. Kolmogorov's and N. V. Smirnov's criteria of fit. Doklady Akad. Nauk SSSR (N.S.) 95, 443–446 (1954). (Russian)

Statement of the results described in the title, with a sketch of the proof. The results of B. V. Gnedenko [same Doklady (N.S.) 82, 661–663 (1952); these Rev. 13, 760] are a special case of the present results. *J. Wolfowitz.*

Blum, Julius R. Multidimensional stochastic approximation methods. Ann. Math. Statistics 25, 737–744 (1954).

Generalizations to functions of several variables of the stochastic approximation methods of (A) Robbins and Monro [same Ann. 22, 400–407 (1951); these Rev. 13, 144] and (B) Kiefer and Wolfowitz [ibid. 23, 462–466 (1952); these Rev. 14, 299, 1278] are shown to have the desired strong convergence properties. In case (A), K families $\{Y_x^{(i)}\}$ $(1\leq i\leq K)$ of random variables are considered for $x \in R^K$. For the corresponding d.f.'s $F_x^{(i)}$ let

$$M^{(i)}(x)=\int y d_y F_x^{(i)}(y).$$

Under certain regularity conditions an approximation scheme generalizing that of (A) is shown to converge strongly to a solution x of $M^{(i)}(x)=\alpha_i$ $(1\leq i\leq K)$. In case (B) for $x \in R^K$, $\{Y_x\}$, F_x, $M(x)$ are similarly considered and, under fairly general conditions (which are much more satisfactory for applications than those assumed in examples where the generalization of (A) is shown to hold), an approximation scheme generalizing that of (B) is shown to converge strongly to the value x, where M is a maximum.

J. Kiefer (Ithaca, N. Y.).

Hildreth, Clifford. Point estimates of ordinates of concave functions. J. Amer. Statist. Assoc. 49, 598–619 (1954).

Let Y_{nt} $(1\leq t\leq T_n,\ 1\leq n\leq N)$ be independent and normal with common unknown variance and $EY_{nt}=\eta_n=\phi(z_n)$, where the z_n are known real numbers and ϕ is only known to be concave. Subject to the latter restriction, the author considers the computation of the maximum likelihood estimator of the vector $(\eta_1, \cdots, \eta_N)$, which involves minimizing $\sum T_n(\eta_n-\bar{Y}_n)^2$ subject to the same restriction (here $T_n\bar{Y}_n=\sum_t Y_{nt}$). Treating this as a problem in nonlinear programming [H. W. Kuhn and A. W. Tucker, Proc. Second Berkeley Symposium Math. Statistics and Probability, 1950, Univ. of California Press, 1951, pp. 481–492; these Rev. 13, 855], the author gives an iterative method for obtaining a solution, and proves its convergence. An application to an agricultural problem is treated, and comments are given on the extension to the case where the z_n are 2-vectors. *J. Kiefer* (Ithaca, N. Y.).

Roy, S. N. Some further results in simultaneous confidence interval estimation. Ann. Math. Statistics 25, 752–761 (1954).

Let Σ_1 and Σ_2 denote the dispersion matrices of two p-variate normal populations. Simultaneous confidence bounds are gotten for all the characteristic values of Σ, and of $\Sigma_1\Sigma_2^{-1}$. Consider a $(p+q)$-variate normal population $(p\leq q)$, and let Σ_{11}, Σ_{22}, Σ_{12} stand respectively for the dispersion submatrix of the first p variates, the last q, and that between the p-set and the q-set. Simultaneous confidence bounds are gotten for all bilinear compounds of $\beta=\Sigma_{12}\Sigma_{22}^{-1}$, the $p\times q$ matrix of regression of the p-set on the q-set. All bounds have confidence coefficients greater than or equal to a preassigned level. This paper is a continuation of another by the author and R. C. Bose [same Ann. 24, 513–536 (1953); these Rev. 15, 726]. *S. W. Nash.*

Malmquist, Sten. On certain confidence contours for distribution functions. Ann. Math. Statistics 25, 523–533 (1954).

Let $X(t)$ $(0\leq t\leq 1)$ be a continuous Gaussian process with $EX(t)=0$ and $E[X(s)X(t)]=\min(s,t)-st$. The author uses Doob's method [same Ann. 20, 393–403 (1949); these Rev. 11, 43] to obtain expressions for such probabilities as

$$P\{X(t)\leq(a-b)(t-\tfrac{1}{2})\ \mathrm{sgn}\ (\tfrac{1}{2}-t)+\tfrac{1}{2}(a+b)\},$$
$$P\{|X(t)|\leq at+b \text{ for } c\leq t\leq d\}, \text{ etc.}$$

These results may be used (asymptotically) to obtain nonparametric one- and two-sided confidence intervals on and tests of hypotheses about continuous distribution functions. Inequalities on the power function for certain alternatives in one such application are considered. The joint distribution of max $X(t)$ and the value t at which this maximum is achieved is also given. *J. Kiefer* (Ithaca, N. Y.).

Jaeckel, K. Statistische Prüfverteilungen endlicher Spannweite. Z. Angew. Math. Mech. 34, 190–191 (1954).

The author proposes the following principle for fitting a probability density of a, say, two-parametric family $\psi(x;\alpha,h)$ to a given probability density $\varphi(x)$: since by Schwarz's inequality

$$\int_{-\infty}^{+\infty}[\varphi(x)]^{1/2}[\psi(x;\alpha,h)]^{1/2}dx \leq \int_{-\infty}^{+\infty}\varphi(x)dx\int_{-\infty}^{+\infty}\psi(x;\alpha,h)dx=1,$$

and equality holds only for $\varphi(x)=\psi(x;\alpha,h)$, he defines those values α, h, which maximize $\int_{-\infty}^{+\infty}[\varphi(x)]^{1/2}[\psi(x;\alpha,h)]^{1/2}dx$ as yielding the best fit. This principle, not claimed to have a probabilistic interpretation, is then applied to a number of concrete examples, in particular to a problem of fitting a distribution of finite range. *Z. W. Birnbaum.*

Bartholomew, D. J. Note on the use of Sherman's statistic as a test for randomness. Biometrika 41, 556–558 (1954).

Hodges, J. L., Jr., and Lehmann, E. L. Matching in paired comparisons. Ann. Math. Statistics 25, 787–791 (1954).

The authors prove the theorem that if X and Y are independent observations on the same unimodal random variable, then $X-Y$ is unimodal. This result is used when they study the test of the effect of a treatment on paired subjects. The test is based on the one-sided sign test. They consider the case where there is a simple alternative, namely, the treatment has a specified positive effect. It is then pointed out that in certain situations it is possible to improve the power of the test by purposely mismatching the paired comparisons. *M. Muller* (Ithaca, N. Y.).

Tsao, Chia Kuei. A simple sequential procedure for testing statistical hypotheses. Ann. Math. Statistics 25, 687–702 (1954).

Let $\{f(x)\}$ be the class of all continuous probability distribution functions defined over a space S. The author suggests the following simple sequential test of $f(x)=f_0(x)$. Divide S into three mutually exclusive sets (zones). S_1 is the zone of preference for acceptance of the null hypothesis, S_2 is the zone of indifference, and S_3 is the zone of preference for rejection. Random observations are drawn successively. At each stage the number of observations falling in each of the three zones will be counted. Let m_i be the number of observations falling in the zone S_i for $i=1, 2, 3$ at the mth stage. Let a and r be two predetermined positive integers. Continue to draw observations as long as $m_1<a$ and $m_3<r$. The experiment is discontinued as soon as $m_1=a$ or $m_3=r$. The null hypothesis is accepted if $m_1=a$ and rejected if $m_3=r$.

Assuming that one will know how to determine the constants a and r and the zones S_1, S_2 and S_3, the author gives implicit formulae for the distribution of the sample size, the moment generating function, the power of the test, and the ASN (average sample number) function. For preassigned a and r the author introduces criteria in order to facilitate the selection of the zones S_1, S_2, and S_3. The existence of zones for any a, r, type-one error α, and power φ is considered. Some possible applications are discussed. For a special case the author approximates numerically the relative efficiencies of his test to that of the Wald sequential probability ratio test. *M. Muller* (Ithaca, N. Y.).

Mitra, Sujit Kumar. A note on minimum variance in unbiased estimation. Sankhyā 14, 53–60 (1954).

The extension of the Cramér-Rao lower bound for the variance of an unbiased estimate given by Chapman and Robbins [same Ann. 22, 581–586 (1951); these Rev. 13, 367] is here applied to some examples. Also a theorem is proven giving a comparison with the Cramér-Rao inequality. [Reviewer's note: Sharper results for the examples involving non-regular estimation can be obtained by a direct attack using the theory of complete sufficient statistics [cf. Lehmann and Scheffé, Sankhyā 10, 305–340 (1950); these Rev. 12, 511]. Also the theorem noted was proved in the paper of Chapman and Robbins cited above.]

D. G. Chapman (Oxford).

Birnbaum, Allan. Combining independent tests of significance. J. Amer. Statist. Assoc. 49, 559–574 (1954).

Let $t_1, \cdots, t_k$ be independent statistics used for testing certain overlapping hypotheses $H_{10}, \cdots, H_{k0}$, large values of the t_i's being significant. Let u_i denote the significance level corresponding to t_i, i.e. the probability, under H_{i0}, of a larger value than the observed one. The u_i's are independent random variables, each of them being uniformly distributed over (0, 1) under the corresponding null hypothesis. The problem is to construct a criterion for $H_0=\bigcap H_{i0}$ based on the u_i's. The present article is an informal but instructive discussion of the question, particularly of the combination methods proposed by Fisher, K. Pearson, and Wilkinson.

G. Elfving (New York, N. Y.).

Méric, Jean. Etude de la formule de Walker donnant le fonction "O.C." du test binomial de Wald. C. R. Acad. Sci. Paris 239, 1117–1119 (1954).

La présente note a pour but de préciser la formule donnée par A. M. Walker [J. Roy. Statist. Soc. Ser. B. 12, 301–307 (1950); ces Rev. 14, 569] sous une forme non explicite, et de rendre l'expression obtenue plus aisément utilisable pour le calcul numérique, grâce à un changement de variables. (Author's summary.) *J. Kiefer* (Ithaca, N. Y.).

Chowdhury, S. B. The most powerful unbiased critical regions and the shortest unbiased confidence intervals associated with the distribution of classical D^2-statistic. Sankhyā 14, 71–80 (1954).

The classical Neyman-Pearson lemma is used to obtain a most powerful unbiased critical region of size α (shortest unbiased confidence interval of confidence coefficient $1-\alpha$) for testing a simple hypothesis concerning Δ^2, the population analogue of Mahalanobis' D^2 statistic, in the case where the two populations have unequal but known covariance matrices. Tables are given. *H. Teicher* (Lafayette, Ind.).

van Klinken, J., and Prins, H. J. Survey of testing and estimation methods with respect to the Poisson distribution. Math. Centrum Amsterdam. Statist. Afdeling. Rep. S 133, 77 pp. (1954). (Dutch)

This useful expository monograph has three chapters. In the first it is supposed that $n=\sum_{i=1}^{m} n_i$ integer observations x_{ij} $(i=1, 2, \cdots, m;\ j=1, 2, \cdots, n_i)$ each derive from a Poisson universe, i.e. the probability of a value x in the (i, j) universe is $e^{-\mu_{ij}}\mu^x_{ij}/x!$ $(x=0, 1, 2, \cdots;\ \mu_{ij}>0)$. Several hypotheses about the μ_{ij} are tested against broader alternatives and the distributions (exact and asymptotic) of the test criteria are considered. The second chapter reviews methods of testing whether n observations x_j are from a (single) Poisson universe or from (a) a binomial universe, (b) a Neyman Type A universe or (c) a compound Poisson universe of specified type, respectively. The final chapter discusses the point and confidence-interval estimates of the Poisson parameter. *H. L. Seal* (New York, N. Y.).

Walsh, John E. Analytic tests and confidence intervals for the mean value, probabilities, and percentage points of a Poisson distribution. Sankhyā 14, 25–38 (1954).

The title very adequately summarizes the paper. The significance tests considered are based on appropriate approximate confidence intervals. The author discusses some

of the approximations used in determining the confidence intervals. The procedure adopted for one-sided confidence intervals consists in presenting results for three different situations: (1) An upper bound for the confidence coefficient is specified; then a lower bound is determined by the values of the sample. (2) A lower bound for the confidence coefficient is specified while the upper bound is determined by the sample. (3) The midpoint between bounds of the confidence coefficient is specified while both bounds are determined by the sample values. These cases then yield corresponding two-sided confidence intervals. *M. Muller.*

Moriguti, Sigeiti. Confidence limits for a variance component. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **3**, 29–41 (1954).

The author obtains useful approximate upper and lower confidence limits for v in the analysis of variance where the expected mean square between rows is σ^2+nv and for error is σ^2. For the case of 90% confidence (two-sided limits) tables are given to aid in the calculations of these limits for $f_1=1(1)6(2)10, 12, 15, 20, 30, 60, \infty$, and $f=6(2)12, 15, 20, 30, 60$ where f_1 is the degrees of freedom for the between rows variance and f is the degrees of freedom for the error term. Comparisons are made with the previous results of Bross [Biometrics **6**, 136–144 (1950)] and Tukey [ibid. **7**, 33–69 (1951)]. *L. A. Aroian* (Culver City, Calif.).

Andrews, Fred C. Asymptotic behavior of some rank tests for analysis of variance. Ann. Math. Statistics **25**, 724–736 (1954).

For testing the equality of c probability distributions on the basis of c independent random samples, Wallis and Kruskal [J. Amer. Statist. Assoc. **47**, 583–621 (1952)] proposed a test based on the statistic $H=A\sum n_i(\bar{R}_i-(N+1)/2)^2$, and Mood and Brown [A. Mood, Introduction to the theory of statistics, McGraw-Hill, New York, 1950; these Rev. **11**, 445] proposed a test based on $M=B\sum n_i(m_i/n_i-[N/2]/N)^2$, where n_i is the size of the ith sample, $\bar{R}_i$ is the average rank of the members of the ith sample after ranking all $N=\sum n_i$ observations, m_i is the number of observations in the ith sample which are less than the median of all observations, and A and B depend on N only. Under the assumption that the ith distribution function is $F(x+\theta_i n^{-1/2})$, $n=\text{const.}\times N$, and certain regularity conditions are satisfied, it is shown that as $N\to\infty$ so that the ratios n_i/n_j remain fixed, both H and M have asymptotic non-central χ^2 distributions with $c-1$ degrees of freedom, with different noncentrality parameters. The asymptotic relative efficiencies (in Pitman's sense) of the H-test, the M-test, and the standard F-test are obtained. *W. Hoeffding* (Chapel Hill, N. C.).

Bhapkar, V. P. A note on t test for paired samples. Calcutta Statist. Assoc. Bull. **5**, 142–147 (1954).

When two series of observations are paired, one of the first series corresponding to each of the second series, two procedures are used to test whether the differences of means is zero: (1) consider the differences of paired values and use Student's t-test, and (2) consider the two series as independent and use a Student's t with twice as many degrees of freedom. Assume equal variances for both series and a constant covariance $\rho\sigma^2$ between pairs. The error of the first kind for the second test varies with ρ, and, for the numerical values of ρ and sample size considered, decreases as ρ increases. Similarly the power of the test decreases as ρ increases. This would indicate that the first test is preferable to the second when $\rho>0$. *S. W. Nash.*

Chernoff, Herman, and Lehmann, E. L. The use of maximum likelihood estimates in χ^2 tests for goodness of fit. Ann. Math. Statistics **25**, 579–586 (1954).

A chi square is formed using k groupings, and the expected frequencies in the cells depend on s parameters which are to be estimated. If the estimators of the parameters are efficient, asymptotically normal, and based on the grouped data, then, as is well known, χ^2 has asymptotically the chi square distribution with $k-s-1$ degrees of freedom. On the other hand, if the estimators are based on the ungrouped original data, the asymptotic distribution of χ^2 is the same as that of

$$\sum_{i=1}^{k-s-1} y_i^2+\sum_{i=k-s}^{k-1} \lambda_i y_i^2,$$

where the y_i are independently normally distributed with mean 0, variance 1, and $0\leq\lambda_i\leq 1$ and may depend on the parameters. The $(1-\lambda_i)$ are the characteristic roots of a certain determinantal equation. *S. W. Nash.*

Putter, Joseph. Sur une méthode de double échantillonnage pour estimer la moyenne d'une population laplacienne stratifiée. Rev. Inst. Internat. Statistique **19**, 231–238 (1951).

The author considers estimation from a stratified normal population. The fraction of the population belonging to each stratum, as well as the total sample size, is assumed to be known while the variance of the estimated character within each stratum is unknown. A double sampling procedure is proposed and an estimate of the variance within strata is obtained from a preliminary sample. The author gives an optimal linear decision rule for allocating the rest of the sample among the strata. A rule is said to be linear if the total size N_h of the sample drawn from the hth stratum is proportional to a multiple of the corresponding standard deviation estimated from the preliminary sample. *E. Lukacs* (Washington, D. C.).

Aoyama, Hirojiro. A study of the stratified random sampling. Ann. Inst. Statist. Math., Tokyo **6**, 1–36 (1954).

Sarkadi, Károly. On the rule of dualism concerning the Bayes' probability limits of the fraction defective. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 275–286 (1954). (Hungarian. Russian and English summaries)

Sarkadi, Károly. On the a priori beta distribution of fraction defective. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 287–297 (1954). (Hungarian. Russian and English summaries)

These papers are connected with work by Oderfeld [Colloquium Math. **2**, 89–97 (1951); these Rev. **13**, 142] and Steinhaus [ibid. **2**, 98–108 (1951); these Rev. **13**, 854] and deal with the application of Bayes' theorem to statistical quality control. The author discusses various assumptions concerning the a priori distribution of the fraction defective and the agreement between what he calls the prospective (confidence limit) and the retrospective (Bayes) method. *E. Lukacs* (Washington, D. C.).

Székely, Gábor. Ein mit der Qualitätskontrolle zusammenhängender stochastischer Prozess. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 217–222 (1954). (Hungarian. Russian and German summaries)

LeCam, Lucien. Note on a theorem of Lionel Weiss. Ann. Math. Statistics 25, 791–794 (1954).

Decision-theoretic methods are used to prove and extend a theorem of Weiss [same Ann. 24, 273–281 (1953); these Rev. 14, 889] concerning an optimum property of the "generalized" sequential probability ratio test, the argument resting on an unpublished paper of the author.

H. Teicher (Lafayette, Ind.).

Weiss, Lionel. Sequential procedures that control the individual probabilities of coming to the various decisions. Ann. Math. Statistics 25, 779–784 (1954).

For the decision problem with a finite number of states of nature and possible decisions and for which the probability of selecting more than a fixed number N of observations is zero, a class C (involving "higher order" Bayes solutions) is given such that, for any decision procedure R' non-ϵ C, there exists $R \epsilon C$ with the probability of making a favorable (unfavorable) decision using R equal to or greater ($\leqq$) than that under R' and also whose probability of requiring fewer than n observations is $\geqq$ than the probability of the same event under R'. *H. Teicher.*

Konijn, H. S. A further remark on the characterization of minimax procedures. Ann. Inst. Statist. Math., Tokyo 6, 123 (1954).

The author proves the familiar result which was noted in the first sentence of the review of the author's previous paper [same Ann. 4, 103–105 (1953); these Rev. 15, 143].

J. Kiefer (Ithaca, N. Y.).

✱Rasch, G. On simultaneous factor analysis in several populations. Uppsala Symposium on Psychological Factor Analysis, 17–19 March 1953, pp. 65–71. Ejnar Munksgaard, Copenhagen; Almqvist and Wiksell, Stockholm, 1953. 10 Danish crowns; 7.50 Swedish crowns.

van IJzeren, J. The theoretical aspect of least squares. Statistica, Rijswijk 8, 21–45 (1954). (Dutch. English summary)

Dealing with the approach where the variance of parameter estimates is minimized, not the residual variance, the exposition centers on the equivalence with best linear parameter estimates. *H. Wold* (Uppsala).

Luvsanceren, Š. Maximum likelihood estimates and confidence regions for unknown parameters of a stationary Gaussian process of Markov type. Doklady Akad. Nauk SSSR (N.S.) 98, 723–726 (1954). (Russian)

The author gives restrictions on the likelihood function under which, he says, the maximum likelihood (m.l.) estimator will be consistent uniformly in the parameter θ to be estimated. His restrictions require the stochastic convergence of certain functions of the likelihood, uniformly in θ. He then considers a stationary Gaussian Markov process with known variance, and states that a) when the mean is known, the m.l. estimator of the correlation coefficient (between two consecutive observations) is asymptotically normal, with variance cited, b) when the mean is unknown, the m.l. estimators of the mean and correlation coefficient are asymptotically normal and independent, with cited variances. The results are given without proof.

J. Wolfowitz (Ithaca, N.Y.).

Seguchi, Tsunetami, and Ikeda, Nobuyuki. Note on the statistical inferences of certain continuous stochastic processes. Mem. Fac. Sci. Kyūsyū Univ. A. 8, 187–199 (1954).

The following assumptions are made on the stochastic process $x(t)$, $-\infty<t<\infty$. 1) The process is continuous in the mean. 2) The covariance function $r(t, s)$ is known. 3) There exist two linearly independent functions $\varphi(t)$ and $\psi(t)$ such that $r(t, s)=\varphi(t)\psi(s)$ for $t\geqq s$ and $r(t, s)=\varphi(s)\psi(t)$ for $t\leqq s$ and such that $r(t, s)$ satisfies a certain second order self-adjoint differential operator considered by Dolph and Woodbury [Trans. Amer. Math. Soc. 72, 519–550 (1952); these Rev. 14 295]. Based on the above assumptions the authors give the unbiased linear minimum-variance predictions at given time points. (The authors claim this is a generalization of Kolmogorov's theorem [C. R. Acad. Sci. Paris 208, 2043–2045 (1939)] for cases where the process is not stationary.) They give the unbiased linear minimum-variance estimate and its variance in terms of certain integral equations. When the process is Gaussian, they give the usual result that the estimate obtained is equal to the maximum likelihood estimate. *M. Muller.*

Watson, G. S. Extreme values in samples from m-dependent stationary stochastic processes. Ann. Math. Statistics 25, 798–800 (1954).

The limiting distribution of the largest among n successive observations on a sequence of independent and identically distributed random variables is well-known [see, e.g., Cramér, Mathematical methods of statistics, Princeton, 1946, p. 371; these Rev. 8, 39]. The author shows that exactly the same limiting distribution is obtained when the sequence is drawn from certain m-dependent stationary stochastic processes. A sequence of random variables $\{x_i\}$ is called m-dependent if $|i-j|>m$ implies that x_i and x_j are independent. *B. Epstein* (Detroit, Mich.).

Foster, F. G., and Stuart, A. Distribution-free tests in time-series based on the breaking of records. J. Roy. Statist. Soc. Ser. B. 16, 1–13; discussion 13–22 (1954).

Two simple statistics are proposed as distribution-free tests of the randomness of a series of observations. Let d denote the number of upper records minus the number of lower records and let s denote their sum. It is shown that d and s are uncorrelated and asymptotically normal. d provides a consistent test against trend in the mean, s a consistent test against trend in the variance. Sampling experiments indicate that the power of the d-test is greater than that of certain other tests but less than that of rank correlation tests. *S. W. Nash* (Vancouver, B. C.).

Kendall, M. G. Note on bias in the estimation of autocorrelation. Biometrika 41, 403–404 (1954).

The author calculates by direct methods the bias term of order n^{-1} in the expectation of the autocorrelation coefficient (various definitions are considered) for the cases when the variate obeys a first-order autoregression or an arbitrary third-order moving average. *P. Whittle.*

Marriott, F. H. C., and Pope, J. A. Bias in the estimation of autocorrelations. Biometrika 41, 390–402 (1954).

The authors calculate by direct methods the bias term of order n^{-1} in the expectation of the autocorrelation coefficient for the cases when the variate obeys a first-order auto-

regression or a particular moving-average scheme. The calculations are confirmed by sampling investigations.
P. Whittle (Wellington).

Westcott, J. H. The minimum-moment-of-error-squared criterion: a new performance criterion for servo mechanisms. Proc. Inst. Elec. Engrs. Part II. **101**, 471–480 (1954).

The author is interested in developing a workable criterion for judging servo mechanism performance. He feels that the usual frequency response criteria, while giving generally correct rules of thumb, are neither comprehensive and quantitative enough to cope with the special or unusual cases. Further he finds that a criterion based on minimizing the integral of the error squared, namely $S=\int_0^\infty \epsilon^2(t)dt$, tends to over-emphasize the importance of the initial response and generally results in under-damped systems. Instead, the author proposes as a suitable figure of merit the first moment in time of the error squared, that is $W=\int_0^\infty t\epsilon^2(t)dt$, which he finds in agreement with the accepted standards of goodness. Two examples in which S and W are each minimized are compared in support of his contention. A method for evaluating S when the Laplace transform of $\epsilon(t)$ is a rational function has been devised by the reviewer [see "Theory of servomechanisms", edited by H. M. James, N. B. Nichols, and R. S. Phillips, McGraw-Hill, New York, 1947; these Rev. **11**, 517] and by introducing an auxiliary parameter the author is able to reduce the evaluation of W to the above case. It should be remarked that the author does not compare his criterion, which is essentially based on the transient response, to the mean-square-error criterion which minimizes $\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\epsilon^2(t)dt$ and therefore evaluates the average response to the actual input. *R. S. Phillips.*

Gamba, A. Information theory and knowledge: remarks on a paper by D. K. C. MacDonald. J. Appl. Phys. **25**, 1549 (1954).

Vallée, Robert. Quelques aspects de la théorie de l'information. Rev. Gén. Electricité **63**, 698–703 (1954).

Mathematical Economics

Inada, Ken-ichi. Elementary proofs of some theorems about the social welfare function. Ann. Inst. Statist. Math., Tokyo **6**, 115–122 (1954).

The author gives shorter proofs of three theorems from the book "Social choice and individual values" [Wiley, New York, 1951; these Rev. **12**, 624] by K. J. Arrow.
D. Gale (Providence, R. I.).

Jacobs, Walter. The caterer problem. Naval Res. Logist. Quart. **1**, 154–165 (1954).

The Caterer Problem is a paraphrased version of a practical military problem which arose in connection with the estimation of aircraft spare engine requirements. In this paper the problem is put into the form of a linear programming problem, and by several transformations an explicit solution of that linear programming problem is obtained for a special case. (From the author's summary.)
J. Kiefer (Ithaca, N. Y.).

Palásti, Ilona, Rényi, Alfréd, Szentmártony, Tibor, und Takács, Lajos. Ergänzung des Lagervorrates. I. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 187–201 (1954). (Hungarian. Russian and German summaries)

Ziermann, Margit. Ergänzung des Lagervorrates. II. Nachbestellung. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. **2** (1953), 203–216 (1954). (Hungarian. Russian and German summaries).

These papers deal with a version of the inventory problem developed for a specific application. The authors consider a factory which requires spare parts for the operation of its machines and determine an optimal ordering policy for these items. The paper departs from the treatment by Dvoretzky, Kiefer, and Wolfowitz [Econometrica **20**, 187–222, 450–466 (1952); these Rev. **13**, 856; **14**, 301] in two major points. (1) Ordering can occur at any instant so that the stochastic process considered as a continuous time parameter. (2) Instead of minimizing some expectation, it is required that the probability of a breakdown of the production process should not exceed a preassigned level. In the first paper the problem of determining the stock-level for a certain part is discussed while the second paper deals with the question of the optimal size of the replacement order. *E. Lukacs* (Washington, D. C.).

Mathematical Biology

Have

***Woodger, J.-H. Problems arising from the application of mathematical logic to biology.** Applications scientifiques de la logique mathématique (Actes du 2e Colloque International de Logique Mathématique, Paris, 1952), pp. 133–139; discussion, pp. 139–140. Gauthier-Villars, Paris; E. Nauwelaerts, Louvain, 1954. 2,200 francs.

Trucco, Ernesto. Studies in imitative behavior: a generalization of the Rashevsky model; its mathematical properties. Bull. Math. Biophys. **16**, 279–316 (1954).

Rashevsky, N. Topology and life: in search of general mathematical principles in biology and sociology. Bull. Math. Biophys. **16**, 317–348 (1954).

Have

***Rashevsky, Nicolas. Two models: imitative behavior and distribution of status.** Mathematical thinking in the social sciences, pp. 67–104, 419–420. The Free Press, Glencoe, Ill., 1954. $10.00.

Resumé of earlier publications by the author and coworkers with particular reference to the two topics named in the title. *A. S. Householder* (Oak Ridge, Tenn.).

***Coleman, James S. An expository analysis of some of Rashevsky's social behavior models.** Mathematical thinking in the social sciences, pp. 105–165, 420–423. The Free Press, Glencoe, Ill., 1954. $10.00.

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. XIII. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 886–887 (1954).

For part XII see same Bull. (5) **39**, 255–256 (1953); these Rev. **14**, 1000.

TOPOLOGY

Kelly, John B., and Kelly, L. M. Paths and circuits in critical graphs. Amer. J. Math. 76, 786–792 (1954).

A k-chromatic graph is said to be critical if it has no proper subgraph which is k-chromatic. The authors shew that if p nodes are deleted from a critical graph the number of connected components in the remaining graph is at most p^p. Actually it follows at once from their proof that this number is $\leqq p$ for $p\leqq 2$ and $<p!$ for $p>2$. Let $L_k(n)$ be the greatest positive integer with the property that every critical k-chromatic graph of order n contains a circuit of length $\geqq L_k(n)$. The authors prove that $L_k(n)\to\infty$ as $n\to\infty$ for all $k\geqq 3$. They also prove that $\lim\inf L_4(n)/\log^2 n\leqq 3/\log^2 6.75$ and they conjecture that $\lim\inf L_k(n)/\log^2 n<\infty$ for all $k>3$. In a paper which is to appear shortly the reviewer has shown that this is true. Finally, they prove that for any k there exists a k-chromatic graph with no circuits of length $\leqq 5$.

G. A. Dirac (Vienna).

Ringel, Gerhard. Farbensatz für orientierbare Flächen vom Geschlechte $p>0$. J. Reine Angew. Math. 193, 11–38 (1954).

The author continues his investigation regarding the number of different colours required for the colouring of maps drawn on higher surfaces. The principal result proved in this paper is that a map consisting of λ_p mutually adjacent countries can be drawn on a closed orientable surface of genus p for all $p>0$, where $\lambda_p=[\frac{1}{2}(3+(1+48p)^{1/2})]$.

G. A. Dirac (Vienna).

Aczél, J. Bemerkungen zur Realisierung der Hausdorffschen Axiome in abstrakten Mengen. Publ. Math. Debrecen 3 (1953), 183–186 (1954).

L'A. construit "effectivement" dans tout ensemble infini E des topologies vérifiant les axiomes des espaces accessible de M. Fréchet et vérifiant de plus l'axiome suivant: a) Aucun point n'admet un voisinage réduit à ce seul point. Mais l'A. a besoin de l'axiome de choix pour parvenir au même résultat lorsqu'il impose de plus l'axiome de séparation de Hausdorff. De telles topologies n'existent pas lorsque E est fini à moins d'affaiblir à la fois l'axiome de séparation des espaces accessibles et l'axiome a), en les remplaçant respectivement par l'axiome de séparation de Kolmogoroff et par l'axiome suivant: a') Il existe au moins un point n'admettant aucun voisinage réduit à ce seul point. Lorsqu'on fait cette double substitution, de telles topologies sont faciles à construire "effectivement" dans tout ensemble fini E.

A. Appert (Angers).

Hämisch, Werner. Über die Topologie in der Algebra. Math. Z. 60, 458–487 (1954).

Généralisation et systématisation des questions se rapportant aux topologies définies par des moyens purement algébriques sur des ensembles munis de structures algébriques. L'auteur définit et étudie d'abord la notion de congruence sur un tel ensemble E. Les topologies en question sont déduites de structures uniformes pour lesquelles un système fondamental d'entourages est formé par les graphes dans $E\times E$ de congruences sur E. Etude des passages aux sous-ensembles, aux produits, aux quotients et aux complétés.

P. Samuel (Clermont-Ferrand).

Geymonat, Ludovico. Su di un metodo per lo studio di spazi astratti molto generali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 12 (1953), 360–366 (1954).

L'A. considère les espaces abstraits constitués par un ensemble K d'éléments appelés "points" et par une relation de spatialisation (ou de contiguïté) ρ qui fait correspondre à tout ensemble $E\subset K$ l'ensemble ρE des points de K qui sont considérés comme contigus à E. Il appelle relation conjuguée de ρ la relation ρ^* définie par les conditions: 1) $\rho^*0=0$; 2) $\rho^*E=K-(\rho E-E)$ pour $E\neq 0$, 0 désignant l'ensemble vide. Etude des relations entre les axiomes topologiques vérifiés par ρ et ceux vérifiés par ρ^*.

A. Appert (Angers).

Geymonat, Ludovico. Analisi della validità degli assiomi di separazione in uno spazio non-V. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 15, 262–264 (1953).

Suite de la note précédente où les axiomes topologiques envisagés sont ceux de séparation. L'A. en déduit l'exemple d'un espace non (V) vérifiant T_2 et T_4 mais ne vérifiant ni T_1 ni T_3.

A. Appert (Angers).

Suzuki, Jingoro. On uniformities agreeing strongly with the topology. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A. 4, 283–289 (1953).

The concepts of uniformity, completely regular uniformity, strong agreement with the topology, and regular extension are taken in the sense of K. Morita [Proc. Japan Acad. 27, 65–72, 632–636 (1951); these Rev. 14, 68, 571]. H. Freudenthal has defined an extension of a space based on a suitable partial ordering $O\ll P$ defined for some pairs of open sets [Ann. of Math. (2) 43, 261–279 (1942); these Rev. 3, 315]. The author's first theorem relates these concepts as follows. Suppose for every neighborhood V of a point x of the space R there is another neighborhood of x such that $U\ll V$. Let U^* be the family of finite open coverings obtained as finite intersections of the binary coverings $\{P, O^{-1}\}$ where $O\ll P$. Then U^* agrees strongly with the topology of R and Freudenthal's extension coincides with the regular extension [see the second paper of Morita cited above] based on the uniformity U^*. Moreover, if for $O\ll P$ we always have also $O\ll Q\ll P$, then U^* is completely regular. The connection is then regarded from the other end, by showing that from a uniformity agreeing strongly, a relation $O\ll P$ can be constructed, and so forth. Turning to another topic, the author shows that the connection between the k-topology and the topology of uniform convergence on compact sets as set forth by the reviewer [ibid. 47, 480–495 (1946); these Rev. 8, 165] still holds when the uniform structure in Weil's sense there used is replaced by a T-uniformity agreeing strongly with the topology. [The unnamed author of a paper cited is J. R. Jackson; but the date is 1952, not 1852.]

R. Arens (Los Angeles, Calif.).

Aotani, Kiyo. Some remarks on the uniform space. Osaka Math. J. 5, 93–98 (1953).

This paper deals with a kind of "locally" uniform structure, in the sense that for any entourage U and any point s there is another entourage V_s such that for any three points p, q, r, one at least of which is s, if (p, r) and (q, r) are V-neighbors, then (p, q) are U-neighbors. This condition is analyzed. Further, appropriate definitions of Cauchy

"sequence" and completeness are given. Finally, it is proved that a space of this kind is bi-uniformly dense in a complete space of this kind having an isomorphic class of entourages. *R. Arens* (Los Angeles, Calif.).

Witt, Ernst. Über den Auswahlsatz von Blaschke. Abh. Math. Sem. Univ. Hamburg **19**, no. 1–2, 77 (1954).

The author proves a result on the uniform space of non-empty subsets of a compact Hausdorff space. As the author himself points out in an "added in proof", this result is established in some exercises in N. Bourbaki's Topologie générale [Actualités Sci. Ind., no. 1142, Hermann, Paris, 1951, p. 145, ex. 7, p. 166, ex. 4, 5; for a review of the first ed. see these Rev. **3**, 55]. *E. Michael.*

Haupt, Otto, und Pauc, Christian Y. Über die durch allgemeine Ableitungsbasen bestimmten Topologien. Ann. Mat. Pura Appl. (4) **36**, 247–271 (1954).

This memoir is an elaboration of a previous note of the authors [C. R. Acad. Sci. Paris **234**, 390–392 (1952); these Rev. **13**, 728]. The results stated there are now proved and the whole theory is developed in a systematic manner. A consistent change of the notation is to be observed: The basis for differentiation (in the sense of de Possel) [="Ableitungsbasis"] is now designated by **a** and, in order to make the relation to the basis **a** apparent, the notations D-topology etc. are now throughout replaced by the notations **a**-topology etc. In addition to the review quoted above the following may be mentioned: The set of "**a**-interior" points of a set M is again designated by $I(M)$. In general this operator I is not idempotent; but this is the case if **a** is a weak Vitali basis. On the other hand, an example (in §5) shows that the property of I to be idempotent is not characteristic for the weak Vitali basis. If I is not idempotent, then for a given M transfinite iteration of I (according to ordinal numbers of the first or second class) is used to obtain an "**a**-open kernel" of M. Another example (of §5) shows that any given ordinal number of the first or second class can really occur here if the basis **a** is correspondingly chosen. §4 discusses how additional assumptions for the fundamental set influence the **a**-topology.
A. Rosenthal (Lafayette, Ind.).

Skornyakov, L. A. Systems of curves on a plane. Doklady Akad. Nauk SSSR (N.S.) **98**, 25–26 (1954). (Russian)

Let Σ be a system of curves in the euclidean plane P with the following properties: Each curve in Σ is homeomorphic to a straight line or to a circle; two distinct points of P lie on exactly one curve in Σ. The following is stated without proof: no curve in Σ is homeomorphic to a circle. Any curve C in Σ is either "open", i.e. traversing C in either direction leads to infinity (in P); or C is "semiclosed", i.e. its closure is homeomorphic to a circle and is obtained from C by adding exactly one point of P. There are exactly two types of systems Σ: 1) all curves in Σ are open; 2) all open curves in Σ pass through one point z and the closures of the semiclosed curves in Σ are obtained by adding z.
H. Busemann (Göttingen).

Zaubek, Othmar. Über Zusammenhangseigenschaften von Grenzmengen. Math. Ann. **128**, 290–304 (1954).

Conditions are secured under which $\limsup A_n$ and $\liminf A_n$ are connected sets when $\{A_n\}$ is a sequence of subsets of a space S which is a Hausdorff space, normal Hausdorff space or metric space. Several of the theorems generalize well known theorems [H. Hahn, Reelle Funktionen, Tl 1, Akademische Verlagsgesellschaft, Leipzig, 1932, §17.1; F. Hausdorff, Mengenlehre, 3. Aufl., de Gruyter, Berlin-Leipzig, 1935, §29.4; G. T. Whyburn, Analytic topology, Amer. Math. Soc. Colloq. Publ., v. 28, New York, 1942, §I.9; these Rev. **4**, 86]. In these theorems the hypothesis (I) that $\bigcup A_n$ is conditionally compact, is replaced by the hypothesis (I′) that any infinite sequence of elements $\{a_n\}$, $a_n \in A_n$ for $n=1, 2, \cdots$, has a limit point in S; and the hypothesis (II) that each A_n is connected, is replaced by the hypothesis (II′) that for almost all n the closure of the set A_n is connected. For example, in a normal Hausdorff space if $\liminf A_n$ is non null, then (I′) and (II′) imply $\limsup A_n$ is connected. Other theorems provide non-trivial necessary and sufficient conditions that one of the limit sets be connected. These theorems, including a generalization of Zoretti's Theorem [J. Math. Pures Appl. (6) **1**, 1–51 (1905), p. 8], have hypotheses difficult to describe in a review and their statements are omitted. *W. R. Utz.*

Bagley, R. W. On orbital topologies. Quart. J. Math., Oxford Ser. (2) **5**, 169–171 (1954).

Let X be an infinite set, let $f\colon X\to X$ be a function, and let $E(x)$ be the union of the sets $f^{-n}(x)$, $n\geqq 0$. Define the closure A^* of the set $A\subset X$ by $(x \in A^*)\leftrightarrow(x \in A$ or $A\cap E(x)$ is infinite). The author translates various properties of the function f into topological language. The last results describe this topology if f is biuniform and onto.
A. D. Wallace (New Orleans, La.).

Kirkor, A. Antoine phenomena and geometric properties of simple arcs. Bull. Acad. Polon. Sci. Cl. III. **2**, 257–260 (1954).

The author constructs in 3-dimensional Euclidean space E a perfect 0-dimensional set A whose complement is not simply connected, and a simple arc α containing A such that (1) α has finite length and (2) α has a continuously turning oriented tangent almost everywhere in the sense of Baire. He remarks that Antoine's set is not contained in any arc of finite length. These results are preceded by a discussion of imbeddings of arcs in E with respect to the mutual relationships between the properties: (1) wildness, (2) containing of perfect 0-dimensional sets with non-simply-connected complement, (3) possession of a continuously-turning oriented tangent, (4) possession of a continuously turning non-oriented tangent. *R. H. Fox.*

Rattray, B. A. An antipodal-point, orthogonal-point theorem. Ann. of Math. (2) **60**, 502–512 (1954).

Let S_n be an n-sphere. The totality α of sets of $n+1$ mutually orthogonal points on S_n can be regarded as a cycle of dimension $n(n+1)/2$ in the $(n+1)$th cartesian power P of S_n. It is shown that if T is an antipode-preserving map $S_n\to S_n$ and T is the induced map $P\to P$, then T_n intersects α. Hence t carries some set of $n+1$ mutually orthogonal points into another such set. The theorem is proved by using the $(n+1)$-fold symmetric product P' rather than P and showing that in P', the intersection number over the integers mod 2 of a certain pair of cycles is $\neq 0$. *P. A. Smith.*

Weier, Josef. Die Homotopieinvarianz der algebraischen Fixpunktzahl. Math. Z. **61**, 82–93 (1954).

Let M be a compact topological manifold and f a mapping $M\to M$. By a small homotopy, f can be reduced to f' where f' has at most a finite number of fixed points, each having a coordinate neighborhood relative to which f' is a non-

singular linear transformation. Let k be the sum of the indices of these linear transformations (where index $A=\det(A-I)$). It is shown that k depends only on the homotopy class of f. *P. A. Smith* (New York, N. Y.).

Hirzebruch, Friedrich. Über die quaternionalen projektiven Räume. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 301–312 (1954).

It is shown that the quaternion-projective spaces, $P_k(Q)$, have no almost complex structure (compatible with the natural differentiable structure of $P_k(Q)$) except possibly for $k=2$ and 3. The proof rests on Wu's formula for the Pontryagin classes in terms of the Chern classes, for almost complex structures. For $P_k(Q)$ one determines the Pontryagin class by considering the well known fibering of complex projective space $P_{2k+1}(C)$ by 2-spheres S_2, with $P_k(Q)$ as base space. The tangent bundle of $P_{2k+1}(C)$ splits into the bundle of the tangent planes to the fibers and the bundle induced from the tangent bundle to the base by the projection. From known facts about $P_{2k+1}(C)$ and S_2 and from the Whitney duality theorems it follows easily that "the" Pontryagin element of $P_k(Q)$ is $(1+u)^{2k+2}(1+4u)^{-1}$, where u is the usual generator of $H^4(P_k(Q), Z)$. Wu's formula requires the existence of an element $c \in H^*(P_k(Q), Z)$, whose square is $(1-u)^{2k+2}(1-4u)^{-1}$. The element $c_k \in H^{2k}$ must be $\pm(k+1)$ times the fundamental cocycle of the space (Euler characteristic). In other words, the coefficient of x^k in the power series $(1-x)^{k+1}(1-4x)^{-1/2}$ must be $\pm(k+1)$. One computes that this is so only for $k=2$ and 3. A similar computation mod 3 shows that for $k\equiv 1 \pmod 3$ there is no almost complex structure on $P_k(Q)$, regardless of the differentiable structure admitted; this is based on the fact that the Pontryagin classes mod 3 are known to be topological invariants. *H. Samelson* (Ann Arbor, Mich.).

Griffiths, H. B. The fundamental group of two spaces with a common point. Quart. J. Math., Oxford Ser. (2) **5**, 175–190 (1954).

Let X_1 and X_2 be closed subsets of $X_1 \cup X_2$ such that $X_1 \cap X_2$ is a point x, the base point for all fundamental groups. It is a corollary of the van Kampen theorem [Amer. J. Math. **55**, 261–267 (1933), not quoted by the author] that $\pi_1(X_1 \cup X_2)$ is the free product of $\pi_1(X_1)$ and $\pi_1(X_2)$ if X_1 and X_2 are locally 0-, 1-, and 2-connected (in the sense of homotopy) at x.

The main theorem of the present paper is that this result holds under the weaker assumption that X_1, say, is locally 1-connected (in the sense of homotopy). That this condition cannot be disregarded is shown by the following startling example: Let $A_\pm$ be the union of the circles

$$(x\mp n^{-1})^2+y^2=n^{-2} \quad (n=1, 2, \cdots)$$

and let $\hat{A}_\pm$ denote the join of $A_\pm$ with the point $(\pm 1, 0, 1)$ of 3-space. Then, although $\hat{A}_+$ and $\hat{A}_-$ are each contractible, hence simply connected, $\pi_1(\hat{A}_+ \cup \hat{A}_-)$ has an uncountable number of elements. (It seems to the reviewer that this last statement can be proved directly without too much trouble; the author makes it look hard by imbedding it in generalizations.) *R. H. Fox* (Princeton, N. J.).

Samelson, H. Groups and spaces of loops. Comment. Math. Helv. **28**, 278–287 (1954).

The author proves that if a topological group G admits a universal bundle E with base B, then corresponding to the contraction of E there exists a continuous map f of G into the space $\Lambda(B)$ of loops in B such that (a): the maps $f(g) * f(g')$ and $f(g\cdot g')$ of $G\times G\to\Lambda(B)$ are homotopic, (b): the maps $f(g^{-1})$ and $[f(g)]^{-1}$ of $G\to\Lambda(B)$ are homotopic, and (c): $f_+ : \pi_i(G) \approx \pi_i(\Lambda(B))$ for all i (and hence the singular homology groups are all isomorphic under f_+). As applications, (1): The map $(x, y)\to xyx^{-1}y^{-1}$, where x, y run through all the quaternions of norm one is not nullhomotopic. (2): The Eilenberg-MacLane spaces $K(Z, n)$ are loop spaces, hence have a Pontrjagin multiplication "$\circ$" for their homologies; it is shown that if z is the generator of $H_{2k-1}(K(Z, n); Z)$, then $z \circ z = 0$. *J. Dugundji.*

Plunkett, Robert L. Representatives of homotopy classes of mappings into spheres. Duke Math. J. **21**, 599–605 (1954).

A continuous mapping f of a continuum M onto a space Y is quasi-monotone if and only if, given any continuum K in Y with non-empty interior, the set $f^{-1}(K)$ has only a finite number of components, and each of these components maps onto K under f. It has been proven by G. T. Whyburn [Duke Math. J. **11**, 35–42, 431–434 (1944); these Rev. **5**, 213; **6**, 164], that every mapping of a locally connected continuum X into the circle S is homotopic to a quasi-monotone mapping of X into S, and that every mapping of a cyclic locally connected continuum X into S is homotopic to an interior mapping of X into S.

The present paper extends these results to the case where the image space is a sphere of arbitrary dimensionality. It is proven that every mapping of a connected polyhedron into a sphere of arbitrary dimension $n\geqq 1$ is homotopic to a quasi-monotone mapping, while a similar theorem holds if the original space is a compact, connected, orientable manifold of dimension m such that $1\leqq m\leqq n$. An example is given, for any dimension $n\geqq 2$, of a locally connected continuum X, and a mapping of X onto the n dimensional sphere which is not homotopic to a quasi-monotone mapping. *D. W. Hall* (College Park, Md.).

Čogošvili, G. S. On spectrally singular homology groups. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 583–588 (1953). (Russian)

The author's aim is to define singular homology groups with compact coefficients [using the "spectral" approach of Hurewicz, Dugundji and Dowker, Ann. of Math. (2) **49**, 391–406 (1948); Rev. **9**, 606]; to this end he gives a modified definition of limit groups: Let $\{A_\alpha, \pi^i{}_{\beta\alpha}\}$ and $\{B_\alpha, \pi_i{}^{\alpha\beta}\}$ be an inverse, respectively direct, system of abelian groups, with the modification that for $\alpha<\beta$ there is a whole family $\{\pi^i{}_{\beta\alpha}\}$, respectively $\{\pi_i{}^{\alpha\beta}\}$, of homomorphisms of the respective groups. The systems are conjugate, i.e., for each α, A_α and B_α are character groups of each other (with duality function $(\ , \ )$ to be reals mod 1), and each $\pi^i{}_{\beta\alpha}$ is the conjugate map to $\pi_i{}^{\alpha\beta}$. The A_α are either all discrete or all compact. The inverse limit A of the A_α is, as usual, the set of those elements $a=(a_\alpha)$ of the direct product $\prod_\alpha A_\alpha$ (discrete if the A are discrete; Tychonoff topology if the A_α are compact), between whose coordinates all relations $a_\alpha=\pi^i{}_{\beta\alpha}a_\beta$ hold. $\prod_\alpha A_\alpha$ is paired with the weak direct product of the B_α by $(a, b)=\sum(a_\alpha, b_\alpha)$; the direct limit B of the B_α is by definition the quotient of the weak direct product of the B_α by the annihilator of A. There is an induced pairing of A and B, by means of which B appears as a set of functions from A to the reals mod 1; B is topologized by the compact-open topology. A is the character group of B; B is dense in the character group of A (and of course equal to it in the case A_α compact); this situation is called

generalized duality. For the application let $L(\mathfrak{A})$ be a vector space of dimension $\mathfrak{A}$ (an infinite cardinality), and consider locally finite simplicial polyhedra P in $L(\mathfrak{A})$, with vertex-sets of cardinality $<\mathfrak{A}$, topologized by J. H. C. Whitehead's weak topology. Let X be a topological space and A a subset. One defines two directed systems Ω_1, Ω_2; each element α is a triple (P, K, f), P one of the polyhedra just mentioned, f a map of P into X; K the subpolyhedron of these simplices which meet A (for Ω_1), respectively are contained in A (for Ω_2). The order $(P, K, f)<(Q, L, g)$ means the existence of an "inclusion isomorphism" s of P into Q with $s(K)\subset L$ and $g\circ s=f$ (it is not clear whether s is supposed to be simplicial). For each $\alpha=(P, K, f)$ in Ω_1 or Ω_2 one constructs now the cohomology groups of P mod K with finite cochains, discrete coefficients, or with infinite cochains, compact coefficients; similarly for homology. The various possible inclusion isomorphisms induce maps of these groups; and with these groups and maps one can now form the limit groups described earlier (using a dual coefficient pair). These limit groups are the new homology and cohomology groups of X mod A, which appear in generalized-dual pairs.

H. Samelson (Ann Arbor, Mich.).

Cartan, Henri. Sur les groupes d'Eilenberg-Mac Lane $H(\Pi, n)$. I. Méthode des constructions. Proc. Nat. Acad. Sci. U. S. A. **40**, 467–471 (1954).

Let Λ be a commutative ring with unit element 1, and let A be a differential graded Λ-algebra with unit element 1, so that Λ is identified with a subalgebra of A. Then an augmentation of A is a Λ-linear map ϵ: $A\to\Lambda$ satisfying:

$$\epsilon(1)=1;\quad \epsilon(xy)=\epsilon x\cdot\epsilon y;\quad \epsilon x=0,\ x\in A_k,\ k>0;\quad \epsilon d=0,$$

where d is the differential. A DGA-algebra is a differential, graded, augmented, anti-commutative Λ-algebra possessing a Λ-basis containing 1 and consisting of homogeneous elements. If A and A' are DGA-algebras, their tensor product over Λ, $A\otimes_\Lambda A'$, is a graded Λ-algebra with unit element and may be given the structure of a DGA-algebra by defining $d''(a\otimes a')=da\otimes a'+(-1)^k a\otimes d'a'$, $a\in A_k$, $\epsilon(a\otimes a')=\epsilon a\cdot\epsilon a'$.

A Λ-linear map f: $A\to A'$ of the DGA-algebra A into the DGA-algebra A' is called a DGA-homomorphism if $f(A_k)\subset A_k'$, $f(1)=1$, $f(xy)=fxfy$, $d'f=fd$, $\epsilon'f=\epsilon$. Then f induces a Λ-linear map f_*: $H_*(A)\to H_*(A')$ which respects multiplication, graduation and the augmentation (induced in $H_*(A)$ by that in A).

A construction (A, N, M) consists of: (i) a DGA-algebra A; (ii) a graded anti-commutative Λ-algebra N having a homogeneous basis consisting of 1 and elements of positive degree; (iii) a differential on $M=A\otimes_\Lambda N$ such that the embedding $A\to M$ given by $a\to a\otimes 1$ respects differentiation and M, with this differential and the augmentation $\epsilon(a\otimes 1)=\epsilon a$, is an acyclic DGA-algebra. A is the initial algebra of the construction, N the final algebra; the projection $M\to N$ given by $a\otimes n\to(\epsilon a)n$ identifies N as a quotient algebra of M and induces in N the structure of a DGA-algebra. [There is a contractible fibre-space, fibred by $K(\Pi, n-1)$ with base-space $K(\Pi, n)$; then we may think of M, A, N as the "algebras" of these spaces. This is, in fact, to be the application.] A homomorphism of (A, N, M) into (A', N', M') is a DGA-homomorphism g: $M\to M'$ which maps A into A'. Then g is said to prolong $g|A$ and induces a DGA-homomorphism $\bar{g}$: $N\to N'$.

A special construction (A, N, M, s) is a construction (A, N, M) together with a Λ-endomorphism s: $M\to M$ with the properties: (i) $dsx+sdx=x-\epsilon x$ (i.e., s is a chain-deformation to the trivial map; recall that M is acyclic); (ii) $s(1)=0$, $s(M_k)\subset M_{k+1}$, $ss=0$; (iii) $1\otimes n\in s(M)$ if $n\in N_k$, $k>0$; (iv) $mm'\in s(M)$ if $m, m'\in s(M)$.

A homomorphism of (A, N, M) into (A', N', M', s') is said to be special if $g(1\otimes n)\in s'(M')$ whenever $n\in N_k$, $k>0$. Then Theorem 1 asserts that a DGA-homomorphism f: $A\to A'$ may be prolonged, in precisely one way, to a special homomorphism, g, of (A, N, M) into (A', N', M', s'), and that $\bar{g}_*$: $H_*(N)\to H_*(N')$ is an isomorphism if f_*: $(H_*A)\to H_*(A')$ is an isomorphism.

The importance of this theorem lies in the fact that the (normalized) bar construction of Eilenberg-Mac Lane [Ann. of Math. (2) **58**, 55–106 (1953); these Rev. **15**, 54] is essentially a special construction in the sense of Cartan. Precisely, let $\bar{A}$ be the quotient Λ-module A/Λ, and put $\bar{\mathfrak{B}}_0(A)=\Lambda$, $\bar{\mathfrak{B}}_k(A)=\bar{A}\otimes\cdots\otimes\bar{A}$ (k times), $k\geqq 1$, $\mathfrak{B}_k(A)=A\otimes_\Lambda\bar{\mathfrak{B}}_k(A)$; then $\bar{\mathfrak{B}}$ is the (normalized) bar construction and, if s: $\mathfrak{B}(A)\to\mathfrak{B}(A)$ is defined by

$$s(a\otimes[a_1]\otimes\cdots\otimes[a_k])=1\otimes[a]\otimes[a_1]\otimes\cdots\otimes[a_k],$$

$(A, \bar{\mathfrak{B}}(A), \mathfrak{B}(A), s)$ is a special construction. Theorem 1 then shows that $H_*(N)\cong H_*(\bar{\mathfrak{B}}(A))$ for any construction with initial algebra A. Moreover, the isomorphism is natural in the sense that commutativity holds round the square

$$\begin{array}{ccc} H_*(N) & \to & H_*(N') \\ \downarrow & & \downarrow \\ H_*(\bar{\mathfrak{B}}(A)) & \to & H_*(\bar{\mathfrak{B}}(A')), \end{array}$$

where N' is the final algebra of a special construction, the vertical arrows are the isomorphisms induced by identity maps, and the horizontal arrows are the homomorphisms induced by some DGA-homomorphism $A\to A'$.

Eilenberg-Mac Lane [loc. cit.] introduced a natural notion of suspension $A_q\to\bar{\mathfrak{B}}_{q+1}(A)$, $q\geqq 1$. Cartan defines a suspension for an arbitrary special construction as the composition $A\to M\xrightarrow{s}M\to N$, and shows that it is natural with respect to maps.

Eilenberg-Mac Lane showed [loc. cit.] that, taking $\Lambda=Z$, $A=Z(\Pi)$, the group ring of an abelian group Π, then $H_*(\bar{\mathfrak{B}}^{(n)}(A))\cong H_*(\Pi, n)$, under a DGA-homomorphism $\bar{\mathfrak{B}}^{(n)}(A)\to K(\Pi, n)$, compatible with suspension, where $\bar{\mathfrak{B}}^{(n)}$ represents the n-fold application of $\bar{\mathfrak{B}}$. Then, if $A'=\Lambda\otimes Z(\Pi)$, it follows that $\bar{\mathfrak{B}}^{(n)}(A')=\Lambda\otimes\bar{\mathfrak{B}}^{(n)}(A)$, so that

$$H_*(\bar{\mathfrak{B}}^{(n)}(A'))\cong H_*(\Pi, n; \Lambda).$$

Thus we have the result that, to calculate $H_*(\Pi, n; \Lambda)$, and the suspension, we may use any n-fold iterated construction starting from $\Lambda(\Pi)=\Lambda\otimes Z(\Pi)$; further, using tensor products, the problem for finitely-generated groups Π may be reduced to the case where Π is a cyclic group of infinite or prime-power order.

This freedom will be exploited in the second note (reviewed below); the power of the method is further enhanced by the fact that the multiplicative structure of the cohomology of $K(\Pi, n)$, with respect to a field of coefficients, is also faithfully represented in any n-fold iterated construction starting from $\Lambda(\Pi)$.

P. J. Hilton.

Cartan, Henri. Sur les groupes d'Eilenberg-Mac Lane. II. Proc. Nat. Acad. Sci. U. S. A. **40**, 704–707 (1954).

In this paper, the author uses the ideas of his earlier note [see the preceding review; the notations are the same] to calculate the homology and cohomology algebras of $K(\Pi, n)$ over Z_p and the homology algebra of $K(\Pi, n)$ over Z. [See also Eilenberg and Mac Lane, Ann. of Math. (2) **58**, 55–106 (1953); these Rev. **15**, 54; and the paper reviewed below.]

Let $E(m;\Lambda)$ be the graduated exterior Λ-algebra with base $(1, x)$, x of degree m, $x^2=0$; x is called the generator. Let $P(m;\Lambda)$ be the graduated "modified polynomial" Λ-algebra with base $(1, x^{(1)}, \cdots, x^{(k)}, \cdots)$, $x^{(k)}$ of degree mk, $x^{(k)}x^{(h)}=(k+h)!(k!h!)^{-1}x^{(k+h)}$; x $(=x^{(1)})$ is called the generator. Let $P^*(m;\Lambda)$ be the graduated polynomial algebra generated by an element x of degree m. If m is even, then $P(m;\Lambda)$ (and $P^*(m;\Lambda)$), furnished with the zero differential and the obvious augmentation, are DGA-algebras; $E(m;\Lambda)$, so furnished, is a DGA-algebra for any m. Then

$$H_*(Z, 1; Z_p)\approx E(1; Z_p)$$

and

$$H_*(Z_{p^f}, 1; Z_p)\approx E(1; Z_p)\otimes_{Z_p} P(2; Z_p).$$

Since Cartan constructions are natural with respect to tensor products, the problem is to obtain constructions whose initial algebras are $E(m-1; Z_p)$, $P(m; Z_p)$, m even. It turns out that there is a construction whose initial algebra is $E(m-1; Z_p)$ and whose final algebra is $P(m; Z_p)$; the generator of $P(m; Z_p)$ is the image under suspension of the generator of $E(m-1; Z_p)$. It further turns out that there is a construction whose initial algebra is $P(m; Z_p)$ and whose final algebra is $\otimes_{k\geq 0} N_k$, where

$$N_k=E(p^k m+1; Z_p)\otimes P(p^{k+1}m+2; Z_p).$$

Suspension sends $x^{(p^k)}$, where x generates $P(m; Z_p)$, to the generator of $E(p^k m+1; Z_p)$, and is zero on other elements of the base.

These two constructions enable the author to give a complete description of the homology algebras $H_*(\Pi, n; Z_p)$ and the suspension homomorphism. Then the cohomology algebras $H^*(\Pi, n; Z_p)$ are deduced, starting from

$$H^*(Z_2, 1; Z_2)\approx P^*(1; Z_2)$$
$$H^*(Z_{p^f}, 1; Z_p)\approx E(1; Z_p)\otimes P^*(2; Z_p),\quad p^f\neq 2,$$

and using the methods of §7 of the earlier note, for $n\geqq 2$. It turns out that the description of $H^*(\Pi, n; Z_p)$ is obtained from that of $H_*(\Pi, n; Z_p)$ by replacing "modified polynomial algebras" by "polynomial algebras".

Similar but more complicated arguments lead to a description of $H_*(\Pi, n; Z)$. Thereby the Bockstein operators, $\beta(p^f)$, which play a vital role in the application of this theory to the calculation of homotopy groups, are determined; in particular, if u_0 is the generator of degree n of $H^*(Z_{p^f}, n; Z_p)$, then $\beta(p^h)u_0=0$, $h<f$, and $\beta(p^f)u_0$ is the generator of degree $n+1$.

Let $\mathcal{P}$: $H^i(X; Z_p)\rightarrow H^{i+2k(p-1)}(X; Z_p)$ be the Steenrod reduced power operation [Proc. Nat. Acad. Sci. U. S. A. **39**, 217–223 (1953); these Rev. **14**, 1006]. The author defines $\mathrm{St}_p{}^a$: $H^i(X; Z_p)\rightarrow H^{i+a}(X; Z_p)$ to be $\mathcal{P}_p{}^\lambda$ if $a=2\lambda(p-1)$ and to be $\beta(p)\mathcal{P}_p{}^\lambda$ if $a=2\lambda(p-1)+1$. He then expresses the generators of the exterior and polynomial algebras which appear in the expression for $H^*(\Pi, n; Z_p)$, $\Pi=Z$ or Z_{p^f}, as iterates of the modified powers $\mathrm{St}_p{}^a$ of the fundamental class u_0. The case $p=2$ was solved by J.-P. Serre [Comment. Math. Helv. **27**, 198–232 (1953); these Rev. **15**, 643]. However, Serre employed a fundamental theorem due to A. Borel on the cohomology mod 2 of fibre-spaces which is not available for arbitrary p.

The stable groups $H_{n+q}(\Pi, n; Z)$, $n>q$, are calculated (they are always torsion groups if $q\geqq 1$). The author verifies that, in the stable case, $H_{n+q}(\Pi, n; Z)\approx H_{n+q}(Z, n; \Pi)$.

P. J. Hilton (Cambridge, England).

Eilenberg, Samuel, and Mac Lane, Saunders. On the groups $H(\Pi, n)$. II. Methods of computation. Ann. of Math. (2) **60**, 49–139 (1954).

In the first paper of the series [Ann. of Math. (2) **58**, 55–106 (1953); these Rev. **15**, 54], it was shown that the complex $K(\Pi, n)$ might be replaced by a complex $A(\Pi, n)$ having a more obvious algebraic structure. In fact, $A(\Pi, n)$ is obtained from $A(\Pi, 0)=Z(\Pi)$ by repeated applications of the bar construction (for notation and definitions see the review of part I), and there is a reduction k_n: $A(\Pi, n)\rightarrow K(\Pi, n)$. In the present paper the authors use the structure of $A(\Pi, n)$ to give invariant descriptions of the homology groups $H_{n+k}(\Pi, n)$, $n\geqq 2$, $k\leqq 5$ except for $H_7(\Pi, 2)$ [see also the two papers reviewed above].

It is obvious geometrically that if $\Pi=\Pi_1+\cdots+\Pi_t$, then there is an equivalence

$$(1)\qquad K(\Pi, n)\rightleftarrows K(\Pi_1, n)\otimes\cdots\otimes K(\Pi_t, n).$$

In chapter I the authors obtain an explicit equivalence

$$A(\Pi, n)\underset{g_A}{\overset{f_A}{\rightleftarrows}} A(\Pi_1, n)\otimes\cdots\otimes A(\Pi_t, n),$$

which is consistent to within a homotopy, under the reductions k_n and $k_n\otimes\cdots\otimes k_n$, with the equivalence (1). This is achieved by exhibiting a contraction

$$(f_B, g_B, \Phi_B):\ B_N(G\otimes G')\rightleftarrows B_N(G)\otimes B_N(G')$$

(B_N = normalized bar construction), based on the Eilenberg-Zilber equivalence $K\times L\rightleftarrows K\otimes L$ for FD-complexes K, L. Diagonal maps

$$e_K:\ K(\Pi, n)\rightarrow K(\Pi, n)\otimes K(\Pi, n),$$
$$e_A:\ A(\Pi, n)\rightarrow A(\Pi, n)\otimes A(\Pi, n)$$

are defined which are again consistent up to a homotopy and lead to a correct description of cup-products in $A(\Pi, n)$.

From the Künneth formula it is clear that

$$H_q(\Pi_1+\Pi_2, n)\cong H_q(\Pi_1, n)+H_q(\Pi_2, n),\quad q<2n.$$

Thus the covariant functor H_* is additive in the stable case. However, it is not additive in general; thus, the Künneth formula shows that

$$H_{2n}(\Pi_1+\Pi_2, n)\cong H_{2n}(\Pi_1, n)+H_{2n}(\Pi_2, n)+\Pi_1\otimes\Pi_2.$$

The extra terms coming in are called cross-effects and it is the presence of such cross-effects, together with other features of the non-stable cases, which creates the main difficulty at this stage and requires the introduction of certain curious functors on the category of abelian groups.

Chapter II discusses the general theory of cross-effects for arbitrary covariant functors on abelian groups. For such a functor T and maps f, g: $A\rightarrow B$, define $T(f\,{}_{\mathsf{T}}\,g)$ as $T(f+g)-Tf-Tg$; then $T(f\,{}_{\mathsf{T}}\,g)$ is a homomorphism $T(A)\rightarrow T(B)$, zero if T is additive. $T(f_1\,{}_{\mathsf{T}}\cdots{}_{\mathsf{T}}\,f_k)$ may be defined similarly. Let $A=A_1+\cdots+A_k$ and let ψ_i: $A\rightarrow A$ be the endomorphism, composition of the projection $A\rightarrow A_i$ with the injection $A_i\rightarrow A$. Then the kth cross-effect functor for T is a covariant functor of k abelian groups; the object function $T(A_1|\cdots|A_k)$ is the subgroup $T(\psi_1\,{}_{\mathsf{T}}\cdots{}_{\mathsf{T}}\,\psi_k)T(A)$ of $T(A)$ and the mapping function $T(f_1|\cdots|f_k)$, for f_i: $A_i\rightarrow B_i$ is the homomorphism $T(A_1|\cdots|A_k)\rightarrow T(B_1|\cdots|B_k)$ induced by $T(f_1+\cdots+f_k)$: $T(A)\rightarrow T(B)$. A functor T is said to be of degree $<k$ if $T(A_1|\cdots|A_k)=0$ (or equivalently $T(f_1\,{}_{\mathsf{T}}\cdots{}_{\mathsf{T}}\,f_k)=0$ for all f_i: $A\rightarrow B$). Explicit examples of functors of degree 2, which are used extensively in the later computations, are given. The functor Γ_* of J. H. C. Whitehead [Ann. of Math. (2) **52**, 51–110 (1950); these Rev. **12**, 43] is of degree 2 and $\Gamma_4(\Pi_1|\Pi_2)\cong\Pi_1\otimes\Pi_2$.

A second functor Ω is related to the torsion product as Γ_4 is related to the tensor product, $\Omega(\Pi_1|\Pi_2)\cong\mathrm{Tor}\,(\Pi_1, \Pi_2)$. To define Ω and obtain this result, as well as for the elucidation of the role of the torsion product in the Künneth formula, the authors find it desirable to give an explicit description of Tor (A, B) in terms of generators and relations. Tor (A, B) is taken to be the abelian group generated by the symbols $\tau_h(a, b)$, $a \varepsilon A$, $b \varepsilon B$, $h>0$, $ha=0$, $hb=0$; the relations are those expressing the bilinearity of $\tau_h(a, b)$, together with

$$\tau_{hk}(a, b)=\tau_h(ka, b),\quad hka=0,\quad hb=0,$$
$$\tau_{hk}(a, b)=\tau_h(a, hb),\quad ka=0,\quad hkb=0.$$

The equivalence of this definition with the usual definition is established.

In Chapter III the computability of $H(\Pi, n)$ for finitely-generated Π is established. This is achieved by replacing $A(\Pi, n)$, where Π is cyclic of order h ($h=2, 3, \cdots, \infty$), by a simpler graded ∂-ring $M(h, n)$ which is, in fact, free and finitely-generated. $M(h, n)$ is defined by iteration of the normalized bar construction from $M(h, 1)$; $M(\infty, 1)$ is an exterior algebra generated by an element of degree 1, and $M(h, 1)$, h finite, is the tensor product of such an algebra with a modified polynomial ring (cf. Cartan, loc. cit.). Moreover, corresponding to a homomorphism $\phi: Z_h\rightarrow Z_{h'}$ a map $\phi_{M,n}: M(h, n)\rightarrow M(h', n)$ is given which is equivalent, up to a homotopy, to $A(\phi, n): A(Z_h, n)\rightarrow A(Z_{h'}, n)$.

A commutative graded ring $\Gamma(\Pi)$ is defined (with $\Gamma_{2n+1}(\Pi)=0$ and Γ_4 the Whitehead functor), with a natural isomorphism of graded rings

$$\Gamma(\Pi_1+\Pi_2)\cong\Gamma(\Pi_1)\otimes\Gamma(\Pi_2)$$

(note: in (18.12) we have $r\neq0$, $s\neq0$). The resemblance of the properties of its elements to those of certain typical cycles of $A(\Pi, n)$ suggests a homomorphism

$$\theta_\Gamma: \Gamma(\Pi)\rightarrow H(\Pi, n),\quad n \text{ even},\tag{2}$$

which multiplies degrees by $n/2$. The structure of $\Gamma(\Pi)$ when Π is cyclic (and hence, by the isomorphism above, when Π is finitely-generated) is obtained; there is an isomorphism $\Gamma(Z_\infty)\cong M(\infty, 2)$.

For the case of n odd, the behavior of certain typical odd-dimensional cycles suggests the definition of the graded exterior ring $\Lambda(\Pi)$. Again we have

$$\Lambda(\Pi_1+\Pi_2)\cong\Lambda(\Pi_1)\otimes\Lambda(\Pi_2)$$

as graded rings, and there is a (1-1) homomorphism

$$\lambda:\ \Lambda(\Pi)\rightarrow H(\Pi, n),\quad n \text{ odd},\tag{3}$$

which multiplies degrees by n. It is proved that, if R is the field of rationals, then

$$\theta_\Gamma:\ \Gamma(\Pi)\otimes R\cong H(\Pi, n; R),\quad n \text{ even},$$
$$\lambda:\ \Lambda(\Pi)\otimes R\cong H(\Pi, n; R),\quad n \text{ odd}.$$

Chapter IV describes the detailed computations of the homology rings $H(\Pi, n)$. If $A_+(\Pi, n)$ is the ideal spanned by all elements of $A(\Pi, n)$ of positive degree, then

$$H_q(\Pi_1, |\cdots|\Pi_t, n)\cong H_q(A_+(\Pi_1, n)\otimes\cdots\otimes A_+(\Pi_t, n)),\quad q>0.$$

from which it follows that

$$H_q(\Pi_1|\cdots|\Pi_t, n)=0,\quad 0<q<nt,$$
$$H_{nt}(\Pi_1|\cdots|\Pi_t, n)\cong\Pi_1\otimes\cdots\otimes\Pi_t;$$

the isomorphisms are made explicit using typical cycles of $A(\Pi, n)$. It is proved that cross-effects suspend to zero.

The known results in low dimensions are given, including the result $H_{n+1}(\Pi, n)=0$, $n>1$, which is clear topologically but is here given an algebraic proof. It is proved that the homomorphism θ_Γ of (2) is (1-1) on the subgroup of $\Gamma(\Pi)$ of elements of degree $\leqq6$ and that it is an isomorphism on this subgroup if $n=2$, i.e.

$$\theta_\Gamma:\ \Gamma_{2t}\cong H_{2t}(\Pi, 2),\quad t=1, 2, 3.$$

If Π is finitely-generated θ_Γ maps this subgroup onto a direct summand of $H(\Pi, n)$ for all even n. The group $\Gamma_8(\Pi)$ is explicitly analysed and $H_8(\Pi, 2)$ is described. The homology rings $H(\Pi, n)$, $n\geqq3$, in dimension $\leqq n+s$ are calculated, together with the suspension, using the constructions and typical cycles defined and discussed earlier.

The last two sections are devoted to the determination, in low dimensions, of the cohomology groups of $K(\Pi, n)$ with coefficients in an arbitrary abelian group G. If Π satisfies $p\Pi=0$, p prime, the authors display a natural isomorphism

$$\rho:\ \text{Extabel}\ (\Pi, G)\cong\text{Hom}\ (\Pi, G/pG).$$

The description of ρ uses the notations and techniques of the present paper, but the proof of isomorphism uses a non-natural description of ρ (an easy proof is available using Extabel $(\Pi, G)=$Hom $(R, G)/\iota^*$ Hom (F, G), where F is free abelian and $F/R=\Pi$).

The descriptions of the cohomology groups require, in the unstable case, a further curious functor $L(\Pi, G)$, defined as the group of all pairs of functions (a, b), where a is a function on Π to $G/2G$ and b is a function on $\Pi\otimes\Pi$ to G. Thus for example,

$$H^6(\Pi, 3; G)\cong L(\Pi, G)+\text{Hom}\ ({}_2\Pi, G),$$

where the second summand is stable.

Two notational innovations are made. A homomorphism $\phi: A\rightarrow B$ is said to be a monomorphism if it is (1-1), an epimorphism if it is onto B, and an isomorphism if it is (1-1) and onto. This last restriction of the use of the term isomorphism is strongly to be recommended. The group of abelian extensions of A by B is written Extabel (A, B), the notation Ext (A, B) signifying the group of central extensions. Errata: p. 53, l. 2, "b_p" should be "b_q"; p. 55, l. 21, "b_p, $b_p \varepsilon G_p'$" should be "b_q, $b_q \varepsilon G_q'$"; p. 59, (4.2), "α, α'" should be "α', α"; p. 88, l. 27, "$[ka_0]=[(k+h)a_0]$" should be replaced; p. 89, l. 1 from bottom, "$\chi_\Lambda(z_1, z_1)+\chi_\Lambda(x_1', z_2)$" should be "$\chi_\Lambda(z_1, z_2)+\chi_\Lambda(z_1', z_2)$"; p. 91, (12.7), "$\chi_\Lambda(z_2, z_1)$" should be "$\chi_\Lambda(z_2, z_1)$"; p. 92, l. 3 from bottom, "$2h(x)$" should be "$2h\gamma(x)$"; p. 117, formulae, replace "x" by "c".

P. J. Hilton (Cambridge, England).

Eilenberg, Samuel, and MacLane, Saunders. On the groups $H(\Pi, n)$. III. Ann. of Math. (2) **60**, 513–557 (1954).

"This paper discusses the role of the groups $H(\Pi, n)$ in the study of the obstruction and classification theorems which are concerned with the secondary obstruction." A full exposition is given of the results announced by Eilenberg [Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 350–353; these Rev. **13**, 575]; the formulations are those of Eilenberg and Mac Lane [Proc. Nat. Acad. Sci. U. S. A. **38**, 325–329 (1952); these Rev. **13**, 966]. The "products" and "depressed products" of the latter paper are here called "operations" associated with elements $\mathbf{y}\ \varepsilon\ H^q(\Pi, n; G)$; they generalize the well-known association of elements

$$\mathbf{y}\ \varepsilon\ H^q(\Pi, n; G)$$

with unary operations $\mathbf{y} \vdash : H^n(K; \Pi) \to H^q(K; G)$ for arbitrary complexes K.

In Chapter I the generalized r-ary operations $\mathbf{y} \vdash$ are defined and their properties discussed. In Chapter II the unary and binary operations are applied to obtain general extension and classification theorems associated with the second obstruction. If Y is a space whose homotopy groups π_i satisfy $\pi_i = 0$, $n < i < q$, then Y has a k-invariant

$$\mathbf{k}^{q+1} \in H^{q+1}(\pi_n, n, \pi_q).$$

Let L be a subcomplex of K and let $f: K^n \cup L \to Y$ map K^{n-1} to the base-point y_0. Then f determines a cochain $a^n(f) \in C^n(K; \pi_n)$ which is a cocycle if and only if f may be extended to $K^{n+1} \cup L$ and thence to $K^q \cup L$. If f admits such an extension, a second obstruction is met whose cohomology class, $\mathbf{z}^{q+1}(f) \in H^{q+1}(K, L; \pi_q)$ depends only on f. The main obstruction theorem then expresses $\mathbf{z}^{q+1}(f) - \mathbf{z}^{q+1}(g)$ by means of unary and binary operations $\mathbf{k}^{q+1} \vdash$; precisely,

$$\mathbf{z}^{q+1}(f) - \mathbf{z}^{q+1}(g) = \mathbf{k}^{q+1} \vdash \mathbf{a}^n(f, g) + \mathbf{k}^{q+1} \vdash (\mathbf{a}^n(f, g), \mathbf{a}^n(g)),$$

where $\mathbf{a}^n(f, g)$ is the class, in $H^n(K, L; \pi_n)$, of $a^n(f) - a^n(g)$ and $\mathbf{a}^n(g)$ is the class, in $H^n(K; \pi_n)$, of $a^n(g)$; here f and g are, of course, two maps $K^n \cup L \to Y$, sending K^{n-1} to y_0, extendable to $K^q \cup L$, and agreeing on L. The homotopy classification theorem is derived from this by the usual device of considering $I \times K$. The special case in which $\pi_i = 0$, $i < n$, gives rise to a formula comprehending the results of Steenrod ($Y = S^n$, $q = n+1$) and others.

In Chapter III the authors return to a study of operations. Operations $P = P(\Pi, G, q, n)$ are defined axiomatically to satisfy certain naturality, coboundary, and additivity formulae verified for the operations $\mathbf{y} \vdash$ and it is proved, in fact, that to each such operation P there exists a unique $\mathbf{y} \in H^q(\Pi, n; G)$ with $P = \mathbf{y} \vdash$. It is shown that $\mathbf{y} \vdash$ is additive (that is, $\mathbf{y} \vdash (\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{y} \vdash \mathbf{x}_1 + \mathbf{y} \vdash \mathbf{x}_2$, or, equivalently, $\mathbf{y} \vdash (\mathbf{x}_1, \cdots, \mathbf{x}_r) = 0$, $r > 1$) if $\mathbf{y}$ is the suspension of an element in $H^{q+1}(\Pi, n+1; G)$. Thus, fixing $q = n+2$, the r-ary operations are all trivial, $n > 2$, $r > 1$. The authors use the structure of $H^{n+2}(\Pi, n; G)$ to prove that, if $\mathbf{y} \vdash$ is additive (e.g., if $n > 2$), then $\mathbf{y} \vdash \mathbf{x} = \mathrm{St}_2{}^2\mathbf{x}$ with respect to a pairing $\Pi \otimes \Pi \to G$ associated with $\mathbf{y}$. If $n = 2$, so that $q = 4$, then $\mathbf{y} \vdash (\mathbf{x}_1, \cdots, \mathbf{x}_r) = 0$, $r > 2$, $\mathbf{y} \vdash (\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2 \smile \mathbf{x}_1$, with respect to a pairing $\Pi \otimes \Pi \to G$ associated with $\mathbf{y}$. The case $\mathbf{y} \vdash \mathbf{x}$, $n = 2$, proves troublesome, but under an assumption which is certainly verified if Π is finitely-generated, $\mathbf{y} \vdash \mathbf{x}$ is the Pontryagin square of $\mathbf{x}$ if $\mathbf{x} \in H^2(K; G)$ and the Postnikov square if $\mathbf{x} \in H^1(K; G)$. Thus (with the restrictions mentioned) a complete description of the operations $\mathbf{y} \vdash$ for $\mathbf{y} \in H^{n+2}(\Pi, n; G)$ is given.

The case $\mathbf{y}$ is a cup-product (arbitrary q) is discussed as well as the case $\mathbf{y} = \mathbf{b}^q$, where $\mathbf{b}$ is the basic class in $H^n(\Pi, n; \Pi)$. The paper closes with some remarks on geometrical interpretations of the invariant $\mathbf{k}$.

On p. 544, Theorem 16.1 (b) and (c), "G" should be "Π"; on p. 545, (17.1′) should be $t(s_1 \top s_2) = 0$; on p. 555, last line, "$f^*\mathbf{k}^{n+2}(S^n)$" should be "$f_*\mathbf{k}^{n+2}(S^n)$". *P. J. Hilton.*

GEOMETRY

Bottema, O. Notes on the skew quadrilateral. Amer. Math. Monthly 61, 692–693 (1954).

Finoulst, J. Remarkable identities connected with regular polygons. Simon Stevin 30, 79–89 (1954). (Dutch. French summary)

The regular polygon $\{n/p\}$ has n sides, enclosing the center p times [Coxeter, Regular polytopes, Pitman, New York, 1949, pp. 3, 93; these Rev. 10, 261]. The ratio of its edge to its circum-radius is $2 \sin (p\pi/n)$, which the author denotes by (n, p). He shows that

$$(n, p) = (kn, p)(kn, n+p)(kn, 2n+p) \cdots (kn, (k-1)n+p)$$

and

$$(kn, p)^2 + (kn, n+p)^2 + (kn, 2n+p)^2 + \cdots + (kn, (k-1)n+p)^2 = 2k.$$

These identities were both anticipated by E. W. Hobson [Plane trigonometry, Cambridge, 1925, p. 119 (28) and p. 91 (Ex. 2)]. In the former, Hobson's θ and n are the author's $p\pi/kn$ and k. In the latter, Hobson's α, β, n are the author's $p\pi/kn$, π/k, k (so that Hobson's $n\beta = \pi$).

H. S. M. Coxeter (Toronto, Ont.).

Rabin, Michael. A theorem on regular polygons. Riveon Lematematika 8, 13–15 (1954). (Hebrew. English summary)

No regular n-gon can have all its vertices in a square lattice unless $n = 4$. This is deduced from the fact that the area would be rational and at the same time a rational multiple of $\operatorname{ctn} \pi/n$. *E. G. Straus* (Los Angeles, Calif.).

Thébault, Victor. Geometry of the tetrahedron. Amer. Math. Monthly 61, 699–700 (1954).

Marmion, A. Sur les sphères podaires par rapport à un tétraèdre. Mathesis 63, 222–236 (1954).

Kuiper, N. H. A plane geometry. Simon Stevin 30, 94–105 (1954). (Dutch)

The so-called isotropic plane geometry (a sub-geometry of the affine geometry with one real isotropic point at infinity) is developed here in an elementary synthetic way with applications to the geometry of the triangle.

O. Bottema (Delft).

Lauffer, Rudolf. Über die Struktur der Konfigurationen (10_3). Math. Nachr. 12, 1–8 (1954).

The ten combinatorially possible configurations 10_3 were enumerated by S. Kantor [S.-B. Math.-Nat. Cl. Akad. Wiss. Wien 84, Abt. 2, 1291–1314 (1881)] and H. Schroeter [Nachr. Ges. Wiss. Göttingen 1889, 193–236]. The author has made an improved classification, based on the observation that six of the ten (including the Desargues configuration) can be split into a quadrangle and a quadrilateral, while the remaining four (including the cyclic configuration [Coxeter, Bull. Amer. Math. Soc. 56, 413–455 (1950), p. 427; these Rev. 12, 350]) can be split into two "great rings", each consisting of five points and five lines which can be regarded as a triangle with two Cevians and their feet.

H. S. M. Coxeter (Toronto, Ont.).

***Byušgens, S. S. On quasi-hyperbolic surfaces (on the theory of an enveloping family of surfaces).** Nomografičeskiĭ sbornik [Nomographic collection], pp. 51–55. Izdat. Moskov. Gos. Univ., Moscow, 1951. (Russian)

Certain technical problems have raised the question of finding the envelope of the circular cylinders of equal radius of which the axes are directed along the generators of one kind on a one-sheeted hyperboloid of revolution. The

formulas for this case are derived and it is shown that the characteristic is a biquadratic curve, the intersection of the cylinder with a hyperbolic paraboloid. This leads to the discussion of the general problem of finding the envelope of surfaces moved without change in form such that the characteristic of their envelope also remains unchanged but for its position in space. *D. J. Struik.*

Athen, Hermann. Vektorrechnung auf der Kugelfläche. Math.-Phys. Semesterber. 4, 90–100 (1954).

A spherical vector (Kugelvektor) is a segment of a great circle on the unit sphere of given length and sense. Two spherical vectors are equal when they are on the same great circle, and have the same length and sense. If the center M of the sphere is connected by radii to the beginning and terminal points A and B of the spherical vector $\mathfrak{c}=\overline{AB}$, and $\overline{MA}=\mathfrak{a}$, $\overline{MB}=\mathfrak{b}$, $\mathfrak{a}$ and $\mathfrak{b}$ ordinary vectors, then $\sin \mathfrak{c}=\mathfrak{a}\times\mathfrak{b}$, $\cos \mathfrak{c}=\mathfrak{a}\cdot\mathfrak{b}$. Addition of two spherical vectors $\mathfrak{a}$ and $\mathfrak{b}$ is performed by moving these vectors on their great circles until the beginning of $\mathfrak{b}$ coincides with the terminal point of $\mathfrak{a}$; then $\mathfrak{a}+\mathfrak{b}$ is the spherical vector connecting the beginning of $\mathfrak{a}$ with the terminal point of $\mathfrak{b}$. This addition is not commutative. Every spherical vector can now be expressed as a linear combination of three other spherical vectors. From the formula $\mathfrak{c}=\mathfrak{a}+\mathfrak{b}$ follows immediately the cosine rule of spherical trigonometry. Other applications to this field are given. *D. J. Struik* (Cambridge, Mass.).

Charrueau, André. Sur diverses transformations géométriques. Bull. Sci. Math. (2) 78, 97–128 (1954).

The matrix formulas, with the order of factors the reverse of the standard notation, for the transform (i.e., conjugate in the projective group) of a correlation by a nonsingular collineation or correlation are first calculated in detail. Although the results would be the same without such specialization, the transformed correlation is restricted to be of period two (i.e., a polarity in a quadric or with respect to a linear complex) and the cases are treated separately. The pencil of polarities with base a repeated hyperplane and a nonsingular quadric is described by a symmetric matrix $m=m_1+\lambda m_2$, with m_1 of rank one and m_2 nonsingular. It is proved that if μ, μ_1, $\mu_2\neq\mu_1$ and μ' are the values of λ for the polarities T, T_1, T_2 and $T'=TT_1T_2$, then μ' is related to μ by the cross-ratio equation $(0, \lambda_2, \mu, \mu')=(0, \lambda_2, \mu_1, \mu_2)$, where λ_2 is the parameter of the second singular polarity of the pencil. Appropriate specializations are made when the hyperplane is tangent to the quadric. The products of more than three polarities of the pencil are also studied.
W. Givens (Princeton, N. J.).

Charrueau, André. Sur les transformations projectives. J. Math. Pures Appl. (9) 33, 263–294 (1954).

[Cf. the preceding review.] After developing properties of skew-symmetric matrices of order four in terms of their duals and Pfaffians, the above cross-ratio equality is again obtained, now relative to an arbitrary basis. Properties of linear complexes are further studied in connection with the Plücker coordinate matrices of a line. *W. Givens.*

***Norden, A. P. Èlementarnoe vvedenie v geometriyu Lobačevskogo. [Elementary introduction to the geometry of Lobačevskiĭ.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 248 pp. 5.05 rubles.

The author aims at an introduction into the geometry of Lobačevskiĭ which does not presuppose some knowledge of higher mathematics, as do previous Russian books on this subject [Ya. Uspenskiĭ, Introduction to the non-Euclidean geometry of Lobačevskiĭ-Bolyai, Petrograd, 1922; S. I. Luk'yančenko, Elements of the non-Euclidean geometry of Lobačevskiĭ-Bolyai, Moscow, 1933; B. V. Kutuzov, The geometry of Lobačevskiĭ and elements of the foundation of geometry, Učpedgiz, Moscow, 1950; P. A. Širokov and V. F. Kagan, 1950; these Rev. 13, 611; B. Ya. Bukreev, 1951; these Rev. 14, 581]. After an introduction in which the relation between geometry and human practice, as well as Euclid's parallel postulate are discussed follows a discussion of those axioms of plane geometry which are independent of this postulate. Then come the basic theorems of Lobačevskiĭ geometry, the defect of polygons, in particular the quadrilaterals of Saccheri (here called after Saccheri and Omar Khayyam) and the plane curves of constant curvature. Special chapters are devoted to space geometry, the geometry on the horosphere, hyperbolic trigonometry and the noncontradictory character of Lobačevskiĭ geometry. The last chapter deals with the relation of this geomatry to other fields of mathematics, especially to projective bnd differential geometry and complex function theory. The nook seems to offer a good example of the way in which eoneuclidean geometry can be taught to undergratues.
D. J. Struik (Cambridge, Mass.).

Maier, W. Inhaltsmessung im R_3 fester Krümmung. Ein Beitrag zur imaginären Geometrie. Arch. Math. 5, 266–273 (1954).

An orthoscheme is a tetrahedron 1234 in which the four face-angles 234, 134, 124, 123, and consequently also the dihedral angles along the edges 13, 23, 24, are right angles. Following Wythoff [Nederl. Akad. Wetensch., Proc. 9, 529–534 (1906)], the author denotes the fifteen remaining angles and edges by ordered pairs of six symbols 0, 1, $\cdots$, 5, with the convention $rs=\frac{1}{2}\pi-sr$. In particular, the pairs 01, 05, 45 denote the dihedral angles along the edges 12, 14, 34. In this notation, Lobachevsky's expression for the volume of a hyperbolic orthoscheme is $\frac{1}{4}iS(10, 05, 54)$, where

$$S(\alpha, \beta, \gamma)=A(\alpha, \delta)-A(\beta, \delta)+A(\gamma, \delta),$$
$$A(\epsilon, \delta)=-i[L(\delta-\epsilon)-2L(\delta)+L(\delta+\epsilon)],$$
$$L(s)=-\int_0^s \log\cos\sigma\, d\sigma,$$
$$\delta=\arctan\left[(\cos^2\beta-\cos^2\alpha\cos^2\gamma)^{1/2}/\sin\alpha\sin\gamma\right]$$

[Coxeter, Quart. J. Math., Oxford Ser. 6, 13–29 (1935), p. 23]. The author has found the following geometrical meaning for Lobachevsky's auxiliary angle δ. Let $4'$ be the point at infinity on the edge 24, and $3'$ the foot of the perpendicular from $4'$ to the line 23 (or to the plane 123). Then the given orthoscheme 1234 is cut off (by the plane 134) from the corner 2 of the asymptotic orthoscheme $123'4'$, in which δ appears as the face-angle $13'2$.

Unhappily there are several misprints; e.g., in the formula on page 267 between (3) and (4). Also the author has changed Lobachevsky's $L(s)$ and δ into $\Lambda(s)$ and ι.
H. S. M. Coxeter (Toronto, Ont.).

Kárteszi, Ferenc. The solution of two construction problems in the hyperbolic plane. Mat. Lapok 4, 87–91 (1953). (Hungarian. Russian and English summaries)

The purpose of this paper is to facilitate the reading of a passage in J. Bólyai's Appendix [1832; new edition, Budapest, 1952]. The constructions discussed are (1) construction

of the angle of parallelism corresponding to a given distance of parallelism, (2) construction of the distance of parallelism corresponding to a given angle of parallelism. Stereometric methods are used. *E. Lukacs* (Washington, D. C.).

Lombardo-Radice, Lucio. L'inversione come dualità nei piani su sistemi cartesiani. Ricerche Mat. **3**, 31–34 (1954).

A Cartesian system is a system in which addition is a (not necessarily abelian) group and multiplication has a unit. Also if $a=b$, c are arbitrary, $xa-xb=c$and $-ay+by=c$ have unique solutions. Here equations $y=xm+b$ determine the finite lines of a projective plane along with the lines $x=c$. Geometrically this corresponds to the minor theorem of Desargues with the line at infinity the fixed axis of perspectivity and the end of the y-axis as the fixed center of perspectivity. Here it is shown that the dual of such a plane is given by a similar system whose addition and multiplication are both anti-isomorphic to those of the given system. Duality is also discussed for planes in which the minor theorem of Desargues has a wider validity.

Marshall Hall, Jr. (Columbus, Ohio).

Hall, Marshall, Jr. Correction to "Uniqueness of the projective plane with 57 points." Proc. Amer. Math. Soc. **5**, 994–997 (1954).

G. Pickert found an error in the middle of page 915 of the author's earlier paper [same Proc. **4**, 912–916 (1953); these Rev. **15**, 460], where the equation marked AD should read

$$12+3s+2t+u=24.$$

This error invalidates the rest of the argument. The author has now succeeded in obtaining the same conclusion another way, though this requires three pages of computation. He takes the opportunity to add a Lemma due to R. H. Bruck: If a plane with $n+1$ points on a line has a proper subplane with $m+1$ points on a line, then either $n=m^2$ or $n\geqq m(m+1)$.

H. S. M. Coxeter (Toronto, Ont.).

Lenz, Hanfried. Herleitung von Dimensionsformeln der projektiven Geometrie aus eingeschränkten Verknüpfungsaxiomen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 81–87 (1954).

Basic formulas for dimension, relative dimension and codimension of the join and intersection of subspaces of a finite- or infinite-dimensional projective space are shown to follow from Veblen and Young's A-1, 2, 3, and the assumption that every line has at least two points.

A. J. Hoffman (Washington, D. C.).

Karzel, Helmut. Ein Axiomensystem der absoluten Geometrie. Arch. Math. **6**, 66–76 (1954).

The paper studies the implications of the axioms used by E. Sperner [Arch. Math. **5**, 458–468 (1954); these Rev. **16**, 278]. On the one hand, the author postulates more than Sperner in that the center of the group generated by the reflection in lines is assumed to consist of the identity only; on the other hand, he requires only the existence of two proper pencils containing a given line in contrast to Sperner's three. As a consequence the author's axioms include the so-called "Lotkerngeometrien", i.e. those where each line is incident with the pencil of its perpendiculars; in standard terminology, the only involutions of the plane are elations.

H. Busemann (Göttingen).

***Efimov, N. V. Vysšaya geometriya. [Higher geometry.]** 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 528 pp. 12.10 rubles.

The third edition differs from the second [1949; see these Rev. **7**, 256; **11**, 124] by various corrections and small improvements. Cantor's Axiom, which in the first two editions was stronger than usual, now appears in its usual form. Two new sections have been added to Chapter III: The basic metric relations in the hyperbolic plane; Brief introduction to Riemann's (i.e., spherical and elliptic) geometry. The material on differential geometry has been moved to the end of the book, which (as indicated in the previous review) is very clear, exact, readable, and therefore useful.

H. Busemann (Göttingen).

Convex Domains, Extremal Problems, Integral Geometry

***Bang, Thøger. Some remarks on the union of convex bodies.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 5–11 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

Let a convex body K in a finite-dimensional vector space be covered by finitely many strips (planks, slabs) S_ν. The width $\Delta(S_\nu: K)$ of S_ν relative to K may be defined as the ratio of the width of S_ν to the width of the minimal strip K_ν parallel to S_ν and containing K (these strip widths being measured on any common transversal). The affine plank problem is the conjecture (A): $\sum\Delta(S_\nu: K)\geqq 1$. It implies the euclidean plank problem: $\sum\Delta(S_\nu)\geqq\Delta(K)$, concerning actual widths in a euclidean space, which was solved by Bang a few years ago [Mat. Tidsskr. B. **1950**, 49–53; also Proc. Amer. Math. Soc. **2**, 990–993 (1951); these Rev. **12**, 352; **13**, 769; also cf. W. Fenchel, Mat. Tidsskr. B. **1951**, 49–51 (1951); these Rev. **13**, 863]. Recently D. Ohmann published an erroneous proof of (A) [J. Reine Angew. Math. **190**, 125–128 (1952); these Rev. **14**, 788]. In the present paper Bang discusses some attacks on the affine plank problem and the difficulties encountered, verifies (A) for the case of two covering strips, and proposes some generalizations.

W. Gustin (Bloomington, Ind.).

Pleijel, Arne. Über die Teilung von ebenen konvexen Bereichen durch Sehnen. Math. Scand. **2**, 74–82 (1954).

The following theorem is proved: if the straight line g divides the plane convex domain c in two parts of areas F_1, F_2 and the contour of c in two parts of lengths L_1, L_2, then the inequality $1+2k-q>0$ holds, where $F_1/F_2=k$, $L_1/L_2=q$. The second member of the inequality cannot be replaced by any positive constant. If F and L are respectively the area and length of c, the stronger inequality $1+2k-q>F^2L^{-4}$ is also proved.

L. A. Santaló (Buenos Aires).

Sandgren, Lennart. On convex cones. Math. Scand. **2**, 19–28 (1954).

Let C be a convex cone in E^n with vertex the origin 0 and let C^* be the dual or polar cone of C, i.e. the set of rays making an angle $\geqq\pi/2$ with all rays belonging to C. The author gives some properties of sets of convex cones and their duals. As an application a simple proof of Helly's theorem on sets of convex bodies is given. The following theorem together with its dual is also proved: Let $\{C_A\}$ be a set of closed convex cones in E^n with the property that

the convex hull of any n of the cones is not the whole space E^n. Then there exists a plane which divides no cone C_A. If all the cones C_A are n-dimensional, this plane may be chosen such as to have n preassigned cones C_A on the same side. *L. A. Santaló* (Buenos Aires).

Ohmann, D. Eine lineare Verschärfung des Brunn-Minkowskischen Satzes für abgeschlossene Mengen. Arch. Math. 6, 33–35 (1954).

Given, for $r=1, \cdots, k$ bounded closed sets A_r in Euclidean n-space and positive reals a_r, where $\sum a_r=1$, the linear combination $A=\sum a_r A_r$ denotes the set of the points expressible as sums $\sum a_r x_r$, where $x_r \in A_r$. By the linear sharpening of the Brunn-Minkowski theorem, the author means the following inequality (2): The Lebesgue measure of A is not less than the corresponding linear combination of the measures of the A_r. The inequality (2) and the Brunn-Minkowski theorem itself are false when one of the sets A_r is empty, since A is then empty, and this inconvenient fact has to be remembered in attempting an inductive proof. For this reason, the author establishes (2) only subject to a somewhat artificial condition (1) on the sets A_r. He verifies however, that the resulting proposition still implies the Brunn-Minkowski theorem. *L. C. Young* (Madison, Wis.).

Kuhn, H. W. Contractibility and convexity. Proc. Amer. Math. Soc. 5, 777–779 (1954).

If X is a subset of real n-space, a subset $U \subset X$ is called a support contact of X if U is the intersection of X with a supporting hyperplane of X. The following theorem is proved: "In order that a locally contractible compact set X in n-dimensional Euclidean space be convex, it is necessary and sufficient that the set X be contractible and that every support contact of X be contractible." The proof proceeds by reducing the result to a theorem on two-person zero-sum games which in turn depends on fixed-point theorems for multi-valued functions. The author observes that a considerably longer proof of this result was obtained by Liberman [Mat. Sbornik N.S. **13**(55), 239–262 (1943); these Rev. **6**, 184] which, however, does not require the hypothesis of local contractibility. *D. Gale.*

Hadwiger, H. Zur Zerlegungstheorie euklidischer Polyeder. Ann. Mat. Pura Appl. (4) **36**, 315–334 (1954).

A polyhedron in k-dimensional euclidean space R_k is defined as a finite aggregate of positive or negative closed simplexes. Two polyhedra A and B are called "G-zerlegungsgleich", where G is a group of movements in R_k containing the translation group (notation: $A\sim B$) if $A=\sum_1^n A_r$ and $B=\sum_1^n B_r$, where A_r and B_r are G-congruent polyhedra. The author develops the theory of "G-Zerlegungsgleichheit" in a lucid way. For arbitrary natural numbers m and n, the equation $X+\cdots+X$ (n times) $\sim A+\cdots+A$ (m times) has a solution X, which is unique when G-zerlegungsgleich polyhedra are regarded as the same; this solution is denoted by $X=(m/n)\cdot A$. Thus the polyhedra form a vector space with respect to the rational number field. This leads to the theorem that $A\sim B$ if and only if $\Phi(A)=\Phi(B)$ for every additive, G-invariant polyhedron functional Φ. In the case when $k=3$ and G is the group of all movements, this theorem was proved in a slightly different form by the reviewer [Mat. Tidsskr. B. **1941**, 59–65; these Rev. **7**, 68]. The paper contains further interesting results; for example, it is shown that any additive, G-invariant functional Φ can be written in the form $\Phi(A)=\sum_1^k \chi_\nu(A)$, where $\chi_\nu(A)$ is homogeneous of degree ν with respect to the rational numbers; i.e., if B is homothetic to A in the rational ratio p, then $\chi_\nu(B)=p^\nu\chi_\nu(A)$. *B. Jessen* (Copenhagen).

Gohier, Simone. Sur la rigidité des calottes convexes à bord. C. R. Acad. Sci. Paris **238**, 1859–1861 (1954).

Following the well known method of Herglotz for proving the rigidy of closed convex surfaces, the following theorem is proved. Let c be the simple closed curve boundary of an open convex surface s; if s is tangent to a sphere (or to a cylinder of revolution) along c, then s is determined by its element of arc except for a motion or a symmetry. *L. A. Santaló* (Buenos Aires)

Rembs, Edouard. Déformabilité des calottes convexes à bande sphérique de bord. C. R. Acad. Sci. Paris **239**, 852–854 (1954).

Let Σ denote a convex surface homeomorphic to a disc; the boundary curve of Σ is supposed to lie on a sphere and the tangent planes of Σ along that curve are also tangent planes of that sphere. Grotemeyer [Math. Z. **55**, 253–268 (1952); these Rev. **13**, 984] and Gohier [see the preceding review] proved that two isometric Σ's are congruent if a tangent plane of either Σ passes through the center O of its sphere only if $O \subset \Sigma$. The author announces the rigidity of Σ with respect to infinitesimal deformations into neighbouring Σ's if it has more than one tangent plane through its O. *P. Scherk* (Saskatoon, Sask.).

Algebraic Geometry

Fabricius-Bjerre, F. An elementary (3,1)-transformation. Nordisk Mat. Tidsskr. **2**, 101–109, 136 (1954). (Danish. English summary)

Let x, y be Cartesian coordinates in a real Euclidean plane such that $x^2+y^2=1$ represents a unit-circle. T_1 and T_2 being points on the parabola $y^2=4ax$, let the tangents (resp. normals) at T_1 and T_2 intersect at x, y (resp. x', y'); then

$$x'=2a+\frac{y^2-ax}{a},\quad y'=\frac{-xy}{a}.$$

These equations define a rational transformation whose properties are studied. In Mat. Tidsskr. B. **1952**, 1–13 [these Rev. **14**, 1116] the author deals with the corresponding problem for an ellipse or hyperbola in a Euclidean or non-Euclidean plane. *F. J. Terpstra* (Bandung).

Primrose, E. J. F. Coincidence points of a curve. Tôhoku Math. J. (2) **6**, 35–37 (1954).

The formula $32m-40n+24k$ is derived for the number of coincidence points [Halphen, Oeuvres, t. 2, Gauthier-Villars, Paris, 1918, p. 198] of a curve of order n, class m, with no singularities other than nodes or cusps, and with k cusps. *G. B. Huff* (Athens, Ga.).

Spampinato, Nicolò. Nozioni introduttive alla teoria delle ipersuperficie algebriche di indice n, dell'S_r proiettivo complesso. I, II, III, IV, V, VI, VII, VIII, IX, X, XI. Rend. Accad. Sci. Fis. Mat. Napoli (4) **14** (1946–47), 1–10, 141–150 (1948); **15** (1948), 13–18, 84–87 (1949); **16** (1949), 4–10 (1950); **17** (1950), 41–47 (1951); **18** (1951), 12–17, 259–264 (1952); **19** (1952), 98–103, 198–199 (1953); **20** (1953), 284–292 (1954).

Spampinato, Nicolò. Le catene t-dimensionali e la geometria corpoproiettiva nell'S_r ipercomplesso di 1^a specie sinistro o destro, legato ad un'algebra dotata di modulo e definita in un qualunque corpo numerico. Rend. Accad. Sci. Fis. Mat. Napoli (4) **16** (1949), 10–13 (1950).

Spampinato, Nicolò. Le geometrie fondamentali, che generalizzano le geometrie non euclidee, in un S_r ipercomplesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) **16** (1949), 169–176 (1950).

Spampinato, Nicolò. Corpovarietà e varietà algebriche ed iperalgebriche di un S_r ipercomplesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 114–117 (1951).

Spampinato, Nicolò. Sulle rappresentazioni complesse di una corpo-varietà algebrica o iperalgebrica e delle trasformazioni corpobirazionali di un S_r legato ad una qualunque algebra complessa. Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 230–238 (1951).

Spampinato, Nicolò. Ipersuperficie e t-ipersuperficie di un S_r ipercomplesso. Rend. Accad. Sci. Fis. Mat. Napoli (4) **18** (1951), 192–198 (1952).

Spampinato, Nicolò. Punti fondamentali e semifondamentali di una trasformazione birazionale di un S_r supercomplesso. Ricerche Mat. **2**, 3–25 (1953).

Spampinato, Nicolò. Le curve ellittiche di un S_r supercomplesso. Ricerche Mat. **2** (1953), 204–240 (1954).

In his "Lezioni di geometria superiore" [v. 9, Pironti, Napoli, 1953] the author has extended the classical theory of elliptic and theta functions to algebras of order n over the complex numbers. In this paper, hypercomplex elliptic curves in a hypercomplex affine S_r are defined and studied in relation to their complex projective models. The cases $n=2$, 3 are studied in detail. *G. B. Huff.*

Carbonaro, C. Marletta. I sistemi omaloidici di ipersuperficie dell'S_4, legati alle algebre complesse d'ordine 4, dotate di modulo. Rend. Accad. Sci. Fis. Mat. Napoli (4) **15** (1948), 168–201 (1949).

Balsimelli, Pio. Le corpoproiettività dell'S_1 biduale nelle tre rappresentazioni complesse. Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 213–218 (1951).

Fadini, Angelo. La prima rappresentazione degli S_1 proiettivi legati a due algebre doppie definite nel corpo C[2]. Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 35–40 (1951).

Fadini, Angelo. Le corpoproiettività dell'S_1 proiettivo legato all'algebra A. Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 118–122 (1951).

Fadini, Angelo. La prima rappresentazione dell'S_1 proiettivo legato ad un'algebra doppia definita nel corpo C[3]. Rend. Accad. Sci. Fis. Mat. Napoli (4) **17** (1950), 340–342 (1951).

***Severi, Francesco. Some remarks on the topological characterization of algebraic surfaces.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 54–61. Academic Press Inc., New York, 1954. $9.00.

This paper gives conditions of a topological character which are necessary and sufficient for a non-singular surface to be birationally equivalent to (i) a plane; (ii) a non-singular quadric; (iii) a ruled surface of genus $p>1$; (iv) an elliptic ruled surface; (v) a hyperelliptic Picard surface. The conditions concern the linear connectivity and torsion, and also certain properties of a base for curves on the surface. In cases (i)–(iv) $p_g=0$ (a topological condition) and every 2-cycle can be represented as a virtual algebraic curve. The conditions are then stated as properties of a base for 2-cycles on the surface. (Some of the conditions involve the Kronecker index of $C\times C$, where C is a member of this base; this cannot be defined directly unless C is a non-singular irreducible curve, but the conditions can easily be stated in an equivalent form which is permissible.) In case (v) it is necessary to select from a base for the 2-cycles a base for the algebraic 2-cycles, and this is not a purely topological operation. *W. V. D. Hodge* (Cambridge, England).

Conforto, Fabio. Sopra i sistemi lineari di integrali semplici di prima specie con periodi ridotti sopra una varietà di Picard. Arch. Math. **5**, 282–291 (1954).

Siano: V_p ($p>1$) una varietà di Picard; Σ il sistema lineare ∞^{p-1} dei differenziali di Picard di prima specie esistenti su V_p; $u_1, u_2, \cdots, u_p$, p differenziali di Σ tra loro linearmente indipendenti; $\Gamma_1, \Gamma_2, \cdots, \Gamma_{2p}$, $2p$ cicli lineari costituenti su V_p una base rispetto all'omologia; ω_{ij} il periodo del differenziale u_i al ciclo Γ_j. Sia poi Σ_q un sistema lineare ∞^{q-1} estratto da Σ: se i $2p$ periodi di ciascuno di q differenziali individuanti Σ_q si possono esprimere come combinazioni lineari a coefficienti interi (uguali per ciascuno dei q differenziali) di r, e non meno, quantità complesse, un fatto analogo si verifica per ogni q-pla di differenziali linearmente indipendenti di Σ_q; Σ_q dicesi allora un sistema lineare di differenaizli semplici di prima specie con r periodi ridotti, e denotasi con $\Sigma_{q,r}$. È noto come la presenza di $\Sigma_{q,2q}$ si verifichi quando la matrice di Riemann $\omega=\|\omega_{ij}\|$ sia impura e V_p possegga una congruenza ∞^{p-q} di V_q abeliane, costituente a sua volta una varietà di Picard; ma non era mai stata approfondita la possibilità dell'esistenza di sistemi $\Sigma_{q,r}$ (che l'Autore chiama "irregolari") con $r\neq 2q$. L'Autore determina la forma di una matrice di Riemann possedente un $\Sigma_{q,r}$, e dimostra: 1) che per ogni $\Sigma_{q,r}$ si ha $r\geqq 2q$; 2) che qualunque sia ω esistono banalmente dei $\Sigma_{q,r}$ con $r\geqq p+q$; 3) che è effettivamente possibile l'esistenza di $\Sigma_{q,r}$ irregolari non banali (per i quali si abbia cioè $2q<r\leqq p+q-1$). Come l'Autore osserva, l'esistenza di sistemi $\Sigma_{q,r}$ è proprio un carattere della V_p (e non della matrice ω soltanto). *D. Gallarati* (Genova).

Nishimura, Hajime, and Nakai, Yoshikazu. On the existence of a curve connecting given points on an abstract variety. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 267–270 (1954).

Etant donné un nombre fini de sous variétés (U_i) d'une variété algébrique V telle que dim $(U_i)<$dim $(V)-1$, il existe, pour tout entier s tel que max (dim (U_i)) $<s<$dim (V), une sous variété V' de dimension s de V contenant les U_i. On se ramène aussitôt au cas où $s=$dim $(V)-1$. Quand V est projective et que les U_i sont des points, l'assertion est alors conséquence du théorème de Bertini [cf. Matsusaka, mêmes Mem. **26**, 51–62 (1950); ces Rev. **12**, 853]; on passe au cas projectif général par section plane générique, puis au cas d'une variété abstraite V par utilisation d'une variété multiprojective dont V est l'image par application régulière. Dans le cas projectif on peut prendre les U_i simples sur V' s'ils sont simples sur V. *P. Samuel.*

Nagata, Masayoshi. Note on intersection multiplicity of proper components of algebraic or algebroid varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 279–281 (1954).

Soient U et V deux variétés de l'espace affine A^n, et W une composante propre de $U\cdot V$. L'auteur déduit de la formule d'associativité des anneaux locaux des conditions nécessaires et suffisantes pour que la multiplicité d'intersection $i(W;\, U\cdot V)$ soit calculable sans passage au produit et à la diagonale: pour qu'elle soit égale à la multiplicité de l'idéal engendré par les équations de U (resp. U et V) dans l'anneau local de W sur V (resp. A^n), il faut et il suffit que U (resp. U et V) soit intersection complète au voisinage de W. En général $i(W;\, U\cdot V)$ est plus petite que cette dernière multiplicité. *P. Samuel* (Clermont-Ferrand).

Nakai, Yoshikazu. Notes on Chow points of algebraic varieties. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **28**, 125–127 (1954).

The author proves the following results. Let V be a projective variety, k a field of definition for V, and $(M_n)_{n=1,2,\cdots}$ an infinite sequence of independent generic points of V over k. Let k_1 be the primitive subfield of k. Then, for n large enough, the Chow point of V is rational over $k_1(M_1, \cdots, M_n)$. Next, let X be a divisor on V, and $c(X)$ its Chow point. Then $\dim_k c(X)$ is the maximum number of independent generic points of V over k which lie on the divisor X. *C. Chevalley* (New York, N. Y.).

Zariski, O. Interprétations algébrico-géométriques du quatorzième problème de Hilbert. Bull. Sci. Math. (2) **78**, 155–168 (1954).

Le 14-ième problème de Hilbert consiste à montrer que, étant donnés un anneau de polynomes (ou, plus généralement, un anneau d'intégrité intégralement clos de type fini) $R'=k[x_1, \cdots, x_n]$ et un sous corps F (contenant k) de son corps des fractions R', l'anneau $R=F\cap R'$ est de type fini (sur k). L'utilisation de modèles projectifs V' de R' et V de R montre qu'il existe un diviseur positif D sur V tel que R soit l'ensemble des fonctions f sur V telles que $(f)\geqq -iD$ pour i assez grand (c.à.d. n'ayant pas de pole en dehors du support de D); d'où, étant donnés une variété normale V et un diviseur positif D sur V, le problème de montrer que l'anneau $R(D)$ des fonctions f telles que $(f)\geqq -iD$ pour i assez grand est un anneau de type fini. Quand V est une courbe, la solution est conséquence facile du théorème de Riemann-Roch. Dans le cas général la solution est assez simple s'il existe un entier i tel que le système linéaire complet $|iD|$ n'ait pas de point base; celà donnerait la solution s'il était permis de remplacer D par un diviseur linéairement équivalent, mais il n'en est rien. Pour avoir une formulation invariante du problème, on introduit la notion de place de base de $R(D)$: c'est une place p de F telle que, pour tout modèle $V^0\geqq V$ de F et pour tout entier i, le centre de p sur V^0 est point base du système $|iD^0|$ (D^0 désignant un diviseur sur V^0 tel que $R(D)=R(D^0)$). Pour que $R(D)$ soit de type fini, il faut et il suffit qu'il n'admette pas de place de base; l'auteur donne aussi une condition équivalente faisant intervenir la topologie de la surface de Riemann de F. Enfin, au moyen d'un lemme sur les systèmes linéaires sur une courbe, on montre que, étant donnés une surface V, un diviseur D sur V et un point simple P de V, P n'est pas point base de $|iD|$ pour i assez grand; d'où, par le théorème d'uniformisation locale, la solution du problème quand V est une surface en caractéristique 0.

(Note: les lignes 4–9 de la p. 165 semblent erronées, mais c'est sans conséquence pour le reste.) *P. Samuel.*

Lang, Serge, and Weil, André. Number of points of varieties in finite fields. Amer. J. Math. **76**, 819–827 (1954).

Let k denote a finite field, of order q, and $V=V_{n,d,r}$ be an algebraic variety defined over k, in a projective space P^n, having the dimension r and the order d. Then it is proved that the number N of points of V which are in k satisfies the inequality

$$|N-q^r|\leqq (d-1)(d-2)q^{r-1/2}+Aq^{r-1},$$

where $A=A(n, d, r)$ is a constant depending only on n, d, r. This theorem for $r=1$, i.e. if V is a curve, is a reformulation of the Riemann hypothesis in function fields; for $r>1$, the proof of the theorem is carried out by an induction on r, leading to evaluation in two different ways of the number of pairs consisting of a point of V in k and a hyperplane in P^n containing this point.

The theorem above leads at once to an asymptotic result. For, if k_ν is the extension of degree ν over k and $N^{(\nu)}$ is the number of points of V in k_ν, then

$$N^{(\nu)}=q^{\nu r}+O(q^{\nu(r-1/2)}) \quad \text{for} \quad \nu\to\infty;$$

and so $N^{(\nu)}\to\infty$ if $\nu\to\infty$. The same asymptotic behaviour is proved to hold for abstract varieties, and applications are obtained to the analytic zeta function $Z(U)$, associated with such a variety V, defined by $d\log Z(U)/dU=\sum_{\nu=1}^{\infty} N^{(\nu)}U^{\nu-1}$. It is shown that $Z(U)$ has no pole or zero in the circle $|U|<q^{-r}$, has exactly one pole in the circle $|U|<q^{-r+1/2}$, namely a pole of order 1 at $U=q^{-r}$, and that the zeros and poles of $Z(U)$ in the circle $|U|<q^{-r+1}$ are birational invariants of V. Finally, some conjectures are stated concerning the behaviour of $Z(U)$ for $|U|\geqq q^{-r+1/2}$. [In the first formula of p. 823, the κ_n in parentheses has to be replaced by κ_{n+1}.] *B. Segre* (Rome).

Weil, André. Sur les critères d'équivalence en géométrie algébrique. Math. Ann. **128**, 95–127 (1954).

In this paper the author gives detailed proofs of theorems which he has announced in a previous note [Proc. Nat. Acad. Sci. U. S. A. **38**, 258–260 (1952); these Rev. **13**, 867]. As a result, some of the important criteria of linear equivalence of divisors are established within the framework of abstract algebraic geometry. The paper is divided into four sections. In §I the author proves, among other results, the following: Let V^n be an abstract variety and let r be an integer less than n; let W be a variety, let Z be a cycle on $V\times W$ and let M, N be two simple points of W such that the cycle $Z(N)-Z(M)=\mathrm{pr}_V[Z\cdot(V\times(N-M))]$ is defined. Then the set G_a of r-cycles on V which can be written in the above form for all possible W, Z, M and N is a subgroup of the group G of all r-cycles on V. The same group G_a is obtained if we restrict the choice of W either to curves or to Jacobian varieties. Next, V^n is assumed to be complete and without multiple subvariety of dimension $n-1$, and in that case, by taking $r=n-1$, the author defines the algebraic equivalence of divisors on V by taking G_a as the reference group, i.e., by taking G_a to be the group of V-divisors which are algebraically equivalent to zero. The notion of linear equivalence is defined in the usual way, i.e., by taking the group G_l of principal divisors as the reference group; G_l is then a subgroup of G_a. Also, "Bertini's theorems" are discussed in §I, which assert, among other things, the following: If V^n is immersed in a projective space P^N and if

L^{N-n+1} is a generic linear subspace of P^N over the smallest field of definition k_0 of V, then $C=V\cdot L$ is defined and is an absolutely irreducible curve such that its singular locus is the intersection of the singular locus of V by L. Therefore, if V^n has no multiple subvariety of dimension $n-1$, which the author always assumes later on, the "generic curve" C is non-singular and does not pass through the singular locus of V. In §II the author proves the existence of a finite set of divisors D_α on V which are algebraic over k_0 and which have the following properties: (a) D_α are free modulo G_l; (b) if X is a V-divisor which is rational over a field k, and if C is generic over k, then $X\cdot C$ is linearly equivalent to zero on C if and only if X is linearly equivalent to a linear combination of the D_α. This result is "elementary" in the sense that it is independent of the theory of Abelian varieties. In §III the author defines a homomorphism $\bar{\varphi}$ of G into an Abelian variety A, provided a rational mapping φ of V into A is given a priori. The definition of $\bar{\varphi}$ and the fact that $\bar{\varphi}$ defines actually a homomorphism of the factor group G/G_l into A are based on a weak part of "Abel's theorem", according to which every rational mapping of a rational variety into an Abelian variety is a constant. In the rest of this section the author proves, among other things, the following result: If a V-divisor X which is algebraically equivalent to zero is annihilated by all possible $\bar{\varphi}$, then a certain integral multiple mX of X ($m\neq 0$) is linearly equivalent to a linear combination of the D_α. This result is preliminary, but it is not at all "elementary", in the sense that its proof makes use of some of the main results of the theory of Jacobian varieties, including the so-called "Riemann inequality". Then §IV is devoted to prove that the D_α are not only free modulo G_l, but also free modulo G_a. This remark gives a definitive form to the above result, which can not be improved even in the classical case. As a consequence of the results reviewed above, the author derives finally the following corollary: If X has a finite order modulo G_a, and if $X\cdot C$ is defined and is linearly equivalent to zero on C, then X itself is linearly equivalent to zero. Here C is a fixed generic curve on V over k_0. This criterion of linear equivalence is not the most general one, since C is generic over k_0, but it is strong enough to enable one to develop the theory of Picard varieties in abstract algebraic geometry.

J. Igusa (Cambridge, Mass.).

Differential Geometry

Dekker, David. Twisted curves and the mean-value proposition. Amer. Math. Monthly **61**, 607–610 (1954).

As is well known, between any pair of points of a plane differential curve there exists a third point at which the tangent is parallel to the chord joining the pair of points. Since twisted cubics and circular helixes have no tangents parallel to any chord, the forgoing result does not hold generally for twisted curves. Indeed, the author now shows, under quite weak hypotheses regarding smoothness and regularity, that if the mean-value property holds for each chord of a twisted curve then the curve necessarily lies in a plane. *E. F. Beckenbach* (Los Angeles, Calif.).

Pinl, M. The ideal straight lines on the catenary surface and its adjoined surface. Math. Student **22**, 137–139 (1954).

The catenary surface, i.e. the surface generated by rotating a catenary about its directrix, and the adjoined screw surface are minimal surfaces. It is here proved that the two surfaces have common ideal straight lines.

Equation (3) and the second of equations (2) are incorrect. In both cases x_3 on the right hand side should be replaced by $(x_1^2+x_2^2)^{1/2}$. *S. B. Jackson.*

Kerawala, Sulaiman. On the integration of the Darboux-Riccati equation for the general helix. Math. Student **22**, 145–147 (1954).

Serini, Rocco. Risultante e momento risultante delle azioni capillari su un pezzo di superficie. Boll. Un. Mat. Ital. (3) **9**, 235–236 (1954).

These theorems were proved by the author once before [same Boll. (2) **3**, 207–210 (1941); these Rev. **3**, 93]. Here they are stated and proved in tensor language. The first theorem reads $\int_C n^r ds=\int_S a^{\alpha\beta}b_{\alpha\beta}N^r dS$, where S is a portion of a surface bounded by curve C, n^r is the exterior normal to C on S, N^r is the normal to S, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the first and second fundamental forms of S. *A. Schwartz.*

Kreyszig, Erwin. Ein elementarer Beweis des Satzes, dass sich kein Teil der Kugeloberfläche längentreu in die Ebene abbilden lässt. Math.-Phys. Semesterber. **4**, 101–105 (1954).

Two proofs are given, neither using Gauss' Theorema Egregium. One depends on the fact that a length-preserving mapping would carry curves of shortest length into curves of shortest length and would also preserve angles. The second depends only on the first of these facts. It supplies a simple configuration of great circles on the sphere which cannot be mapped onto a plane with preservation of lengths.

A. Schwartz (New York, N. Y.).

Busemann, Herbert. Metrics on the torus without conjugate points. Bol. Soc. Mat. Mexicana **10**, nos. 3–4, 1–18 (1953).

A Spanish translation of this paper has already been reviewed [same Bol. **10**, nos. 1–2, 12–29 (1953); these Rev. **15**, 557].

Jha, P. On the locus of the centre of spherical curvature. Ganita **4**, 131–134 (1953).

Consider the curves $C, C_1, C_2, \cdots, C_n$, where each is the locus of the center of curvature of the preceding one. Let s, $1/\rho$, $1/\theta$ be the arc length, curvature, and torsion of C. The author shows that C_2 and C will be identical if

$$(1)\qquad \frac{\theta}{\rho}\frac{d}{ds}\left(\theta\frac{d\rho}{ds}\right)+\frac{d}{ds}\left[\rho\frac{d}{ds}\left\{\rho+\theta\frac{d}{ds}\left(\theta\frac{d\rho}{ds}\right)\right\}\right]=0.$$

If C has constant curvature, then C_1 has the same constant curvature and C_2 and C are identical while C_1 and C_3 are identical. If C is a helix, $\rho=c\theta$, then a solution of (1) is $\theta=(a+bs)^{1/2}$. Lastly, (1) is solved in the case for which ρ is a linear function of s. $\theta=$constant is not one of these solutions, but if ρ is linear in s and θ is constant, then the same is true for C_2. *A. Schwartz* (New York, N. Y.).

Backes, F. Sur la déformation, due à Bonnet, des surfaces à courbure moyenne constante. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 938–943 (1954).

Darboux [Leçons sur la théorie générale des surfaces, t. III, Gauthier-Villars, Paris, 1894, p. 382] gives a treatment of the theorem of Bonnet on surfaces in three-dimensional Euclidean space of constant mean curvature. This treatment is restricted to the study of the first fundamental form.

The author, by means of the Gauss-Codazzi equations, obtains results regarding the second fundamental form. He states his results in terms of explicit formulas, too complicated to reproduce here. *L. Auslander.*

Prvanovitch, Mileva. Hyperlignes de Darboux appartenant à l'espace riemannien. Bull. Sci. Math. (2) **78**, 89–97 (1954).

Let a Riemannian manifold V_n be imbedded in a V_m. Let a congruence of curves be given in the V_m. Let C be a curve of the V_n with arc length s, $t^\alpha{}_{1|}$ and $t^\alpha{}_{2|}$ as first and second unit normal vectors, and R_1 and R_2 as radii of first and second curvature. If at each point of C the geodesic surface determined by the tangent to C and the vector $R_1t^\alpha{}_{1|}+(dR_1/ds)R_2t^\alpha{}_{2|}$ contains the tangent to the V_m congruence curve at this point, then C is called a hyperline of Darboux. The differential equation for these curves is derived and it is pointed out that the lines of Darboux are included among them. Geodesics of V_n which are also geodesics of V_m are hyperlines of Darboux, as are those curves of V_n for which $\delta q^\alpha/\delta s=0$, where q^α is the curvature vector with respect to the V_m. *A. Schwartz.*

Prvanovitch, Mileva. On Darboux lines in a sub-space of a Euclidean space. Rev. Fac. Sci. Univ. Istanbul (A) **19**, 13–18 (1954).

L'article traite d'une généralisation de la notion de ligne de Darboux pour un espace (de Riemann) V_n plongé dans un espace euclidean E_m. Les lignes de Darboux sont alors celles dont l'hypersphére osculatrice en chaque point est tangente à V_n. L'auteur donne, pour les équations différentielles de ces lignes, des développements où interviennent les coefficients des deux formes fondamentales de V_n et leurs dérivées partielles. *P. Vincensini* (Marseille).

Gheorghiu, Gh. Th. Sur une classe de surfaces. II. Acad. Repub. Pop. Române. Stud. Cerc. Mat. **3**, 499–527 (1952). (Romanian. Russian and French summaries)

[For part I see Bull. Sci. Ecole Polytech. Timişoara **10**, 87–92 (1941); these Rev. **8**, 602.] Let (x) and (x') describe two surfaces which are 1) applicable, 2) asymptotic transforms, 3) such that x' is on the Lie quadric of (x) at x and vice versa, 4) such that the asymptotic tangents xx_u, xx_v intersect respectively the asymptotic tangents $x'x_u'$, $x'x_v'$. Then the author has shown [Disquisit. Math. Phys. **4**, 131–173 (1945); Bull. Sci. Tech. Polytech. Timişoara **13**, 141–155 (1948); these Rev. **8**, 486; **10**, 401] that the surfaces must be either surfaces of Tzitzéica, or surfaces of O. Mayer, or surfaces of a new class called (S_2). Here, select for a tetrahedron of reference the points $x\equiv x_1$, $x'\equiv x_4$, and the points of intersection of the asymptotic tangents $\equiv x_2$, x_3. It is shown that (x) and (x') are homographic transforms, and that the line xx' passes through a fixed point O while the line x_2x_3 lies in a fixed plane P. If O' is the point of intersection of xx' and P, then x, x', O, O' are in harmonic ratio. From the equations of the two Lie quadrics it is shown that the surfaces are surfaces of Tzitzéica if they have the same Lie quadric; otherwise the Lie quadrics intersect in two plane curves, one in the fixed plane P and the other in a variable plane which envelopes a cone with vertex at O. If this variable plane contains the asymptotic tangents xx_u, $x'x_u'$ then the surfaces are surfaces of Mayer, otherwise surfaces S_2. Next the author describes the surfaces from a centro-affine point of view, using the fixed point O, the fixed plane P, the Lie quadric, and also the osculating quadric which has the point O as pole of the plane P. A particular S_2 surface is found: $z(x^2+y^2)$ arc $\tan^2 yx^{-1}=3$. To characterize a surface of the type studied, rather than a pair of such surfaces, the author refers the surface to its asymptotic lines and considers the network which arises upon projection from a fixed point onto a fixed plane. The configuration used depends upon the Lie quadric of the given surface and Laplace transform points with respect to the network. In a last section there are given two more properties of the surfaces (S_2). One deals with the ruled surfaces swept out by the edges of a tetrahedron of reference; the other deals with the central indicatrix curves of O. Mayer. *A. Schwartz* (New York, N. Y.).

Vincensini, Paul. Sur les surfaces dont les réglées asymptotiques d'un système appartiennent à des complexes linéaires. C. R. Acad. Sci. Paris **239**, 1113–1114 (1954).

Asymptotic ruled surfaces of a surface S are generated by the asymptotic tangents of one system of S through the points of the same asymptotic line of the other system. Following L. Godeaux, those surfaces S for which a family of such ruled surfaces belong to linear complexes are called F_2-surfaces. A method is given by which such F_2-surfaces can be found. It is based on the relations

$$\begin{aligned}2p_{24}x_1+p_{14}+p_{23}&=2p_{34}x_2-i(p_{14}-p_{23})\\&=2p_{34}x_3+p_{24}+p_{31}=2p_{34}x_4-i(p_{24}-p_{31})=0,\end{aligned}$$

which define a correspondence between the straight lines p_{ij} of a euclidean space and the points x_i of an E_4, and in which the linear complexes of E_3 are associated with the hyperspheres (or hyperplanes) of E_4. Some specific examples of these F_2 are given, among them all surfaces S which are inverse transforms in the sense of Lie of surfaces of Monge. *D. J. Struik* (Cambridge, Mass.).

Dou, Alberto. Plane four-webs. Mem. Real Acad. Ci. Art. Barcelona **31**, 133–218 (1953). (Spanish)

Greek letters $\sigma_i=p_i(u,v)du+q_i(u,v)dv$, α, β, $\cdots$ denote Pfaffians; $[\sigma_1\sigma_2]=(p_1q_2-p_2q_1)[dudv]$ is the exterior product of σ_1 and σ_2; $d\sigma=[dpdu]+[dqdv]=(q_u-p_v)[dudv]$ is the exterior differential of σ. A plane four-web T^4 is given by four equations $\sigma_i=0$ $[i=1, 2, 3, 4]$. The author's discussion of its properties near a point P is based on the "symmetric representation"

$$(1)\qquad \begin{aligned}\sigma_1&=\alpha\cos r-\beta\sin r, & \sigma_2&=\alpha\sin r+\beta\cos r,\\ \sigma_3&=\alpha\cos r+\beta\sin r, & \sigma_4&=-\alpha\sin r+\beta\cos r.\end{aligned}$$

$\Omega=[\alpha\beta]$ vanishes nowhere and the scalar $r=r(u,v)$ remains $\not\equiv 0$ (mod $\pi/4$). Conversely, (1) determines (locally) a T^4 if these conditions are satisfied. The case $r=$const. yields Thomsen's "Doppelverhältnisgewebe" [Abh. Math. Sem. Hamburg. Univ. **8**, 115–122 (1930)]. If in addition one of the four sub-three-webs of T^4 is hexagonal, then T^4 is homeomorphic to a web of four pencils of parallel straight lines [Chapter I].

The general four-web T^4 determines the three-web T^3: $\alpha=0$, $\beta=0$, $\alpha+\beta=0$. Put $h_\alpha=d\alpha/\Omega$, $h_\beta=d\beta/\Omega$, $\gamma=h_\beta\alpha-h_\alpha\beta$. Then T^3 has the curvature $K=d\gamma/\Omega$. The Pfaffians α, β are determined only up to a common scalar factor $m=m(u,v)$. A scalar function $f=f(u,v)$ is an invariant of T^4 if it is not affected by a change of the parameters u, v and if a renorming $\alpha=m\cdot\alpha^*$, $\beta=m\cdot\beta^*$ transforms it into $f^*=m^p\cdot f$ [$p=$const.$=$weight of f]. Then $f_\alpha'=[df,\alpha]/\Omega+ph_\alpha f$ and

$f_\beta' = [df, \beta]/\Omega + ph_\beta f$ are invariants of weight $p+1$. The values of the "symmetrical invariants"

$$(2) \qquad r, r_\alpha', r_\beta', r_{\alpha\alpha}'', \cdots \quad \text{and} \quad K, K_\alpha', K_\beta', \cdots,$$

at P form a complete set of invariants of T^4 at P. The weights of r and K are 0 and 2 respectively. Suppose $z(u, v)$ is continuously differentiable and $s(u, v)$ and $t(u, v)$ are continuous near P; $z \not\equiv 0 \pmod{\pi/4}$ and $z_u z_v st < 0$ at P. Then there is one and only one T^4 in some neighbourhood Γ of P such that $r = z$, $r_\alpha' = s$, $r_\beta' = t$ in Γ [Chapters II and III].

Suppose T^4 is given in the form $u = u(x, y)$, $v = v(x, y)$, $s = s(x, y)$, $t = t(x, y)$. Thus its four pencils are the curves u=const, etc. Eliminating x and y and solving for s and t, T^4 obtains the representation

$$(3) \qquad s = s(u, v), \quad t = t(u, v).$$

Without loss of generality let $u = v = 0$ at P. If s and t are analytic functions of u and v, they permit expansions $s = \sum_0^\infty a_{ik} u^i v^k$, $t = \sum_0^\infty b_{ik} u^i v^k$. Then (3) represents a T^4 near P if and only if $a_{10} a_{01} b_{10} b_{01} (a_{10} b_{01} - a_{01} b_{10}) \neq 0$. The web T^4 is not affected by transformations $\bar{u} = \bar{u}(u)$, $\bar{v} = \bar{v}(v)$, $\bar{s} = \bar{s}(s)$, $\bar{t} = \bar{t}(t)$. By means of them the canonical expansion is obtained. It satisfies $a_{00} = b_{00} = 0$; $a_{10} = b_{01} = 1$, $-a_{01} = b_{10} < -1$; $a_{n0} = b_{0n} = a_{n-1,1} = b_{1,n-1} = 0$ [$n = 2, 3, \cdots$]. These conditions determine u, v, s, t uniquely up to transformations $u = g \cdot \bar{u}$, $v = g \cdot \bar{v}$, $s = g \cdot \bar{s}$, $t = g \cdot \bar{t}$ where g=const$\neq 0$. The coefficients of the canonical expansion form another complete set of invariants of T^4 at P. Hence they can be expressed in terms of the symmetrical invariants and vice versa. The author actually computes the symmetrical invariants of weights $\leqq 2$ in terms of these canonical invariants [Chapter IV].

In Chapter V the curvatures of the four sub-three-webs of T^4 are expressed through (2). For one of them an elegant expression through the canonical invariants is given.

The rank n of T^4 is the maximum number of linearly independent systems of abelian normal parameters. Blaschke and Bol give a proof that $n \leqq 3$ and discuss the case $n = 3$ [Geometrie der Gewebe, Springer, Berlin, 1938, §27]. The author proves that $n = 2$ if exactly three of the four sub-three-webs of T^4 are hexagonal. If $n \geqq 1$, then a certain ten-rowed determinant must vanish. Its terms involve the canonical invariants of T^4. If $n \geqq 2$ or $n = 3$, similar additional conditions are satisfied. We have $n \geqq 1$ if and only if a certain system of two partial differential equations with one unknown function is solvable [Chapter VI].

P. Scherk (Saskatoon, Sask.).

Blaschke, Wilhelm. Eine Abzaehlung in der Geometrie der Waben. Rev. Fac. Sci. Univ. Istanbul (A) **19**, 28–33 (1954).

Any plane three-web T^3 can be represented by three Pfaffians $\alpha = 0$, $\beta = 0$, $\alpha + \beta = 0$. If K is the curvature of T^3, a complete system of (topological) invariants of T^3 is given by: $K, K_\alpha', K_\beta', K_{\alpha\alpha}, \cdots$ [cf. the paper reviewed above; the above notation is Dou's]. The number of independent invariants of T^3 of order $\leqq n$ is $m_t = n(n-3)/2$. If T^3 consists of three pencils of straight lines, a complete independent system of projective invariants of T^3 is readily constructed. There are $m_p = 3(n-1)$ of them of order $\leqq n$. Thus $m_t \leqq m_p$ for $n \leqq 8$ but not for $n > 8$. The author concludes: "Wir haben zu erwarten, dass man eine beliebige Wabe in der Umgebung achter Ordnung einer Stelle geradlinig machen kann. Dagegen haben wir in neunter Ordnung Bedingungen für die Streckbarkeit einer Wabe zu erwarten."

P. Scherk (Saskatoon, Sask.).

Haimovici, Adolf. Considérations sur les réseaux dans un espace à trois dimensions. Acad. Repub. Pop. Române. Fil. Iaşi. Stud. Cerc. Şti. **4**, 29–52 (1953). (Romanian. Russian and French summaries)

This study on nets in a Riemannian three-space is based on work by J. Dubnov and S. Fuchs [C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 102–105 (1940); these Rev. **2**, 161]. Using the tensor calculus to express the properties of the three congruences L_1, L_2, L_3, conditions are found that the planes determined by L_1, L_2 are displaced parallelly along L_3, that the tangents to the L_1 are displaced parallelly along L_2, that the lines L_1 are geodesic, and that there exist holonomy. Other results are the conditions that the net be a "strong" net, or a "weak" net of Tschebycheff. In a "strong" net the tangents to the curves of one family are parallel along the other families, in a "weak" net the tangent planes to the curves of two families are parallel along the other family.

D. J. Struik (Cambridge, Mass.).

Blanuša, Danilo. Le plan elliptique plongé isométriquement dans un espace à quatre dimensions ayant une courbure constante. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 41–58 (1954). (Serbo-Croatian summary)

Blanuša, Danilo. Über die isometrische Einbettung elliptischer Räume in höhere Räume konstanter Krümmung. Acad. Serbe Sci. Publ. Inst. Math. **6**, 91–114 (1954). Please read the concerning note on p. 1337.

In a previous paper [same Glasnik **8**, 3–23 (1953); these Rev. **14**, 1122] the author gave a constructive imbedding of the elliptic plane El_2 in R_5 in such a way that the geodesics of El_2 become circles of R_5. The imbedding of El_2 in the sphere S_4 is now effected by first imbedding S_4 in R_5 in a special position with respect to El_2, and by projecting El_2 on S_4. The projection Π of El_2 is not isometric to El_2, but one can deform Π in S_4 in such a manner that the result is isometric to El_2 again. The deformation, which involves the use of elliptic integrals, causes the imbedding to be analytic in almost all points of El_2, but the geodesics of El_2 are no longer geodesics or circles of S_4. The imbedding is effected only if the curvature ρ of S_4 satisfies $\rho > r/\sqrt{3}$, where r is the curvature of El_2. The imbedding of El_2 in the hyperbolic space H_4 is obtained by generalization of the imbedding formulas for S_4. The imbedding in R_4, which was previously published by the author [Nachr. Öster. Math. Ges. **6**, no. 21/22, 40 (1952)] follows for $r \to \infty$.

The second paper deals with the same problem, but now for imbeddings of El_m in S_{M-1} ($M = m(m+3)/2$), in H_{M-1}, and in R_{M-1} as the limiting case. For the imbedding in S_{M-1} for m even there is the restriction $\rho > r[m/2(m+1)]^{1/2}$, while for m odd there is no restriction on ρ. These results are obtained in the same way, using the imbedding of El_m in R_M in which the geodesics of El_m become circles in R_M [Blanuša, Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 81–114 (1953); these Rev. **15**, 467].

A. Nijenhuis (Princeton, N. J.).

Kurita, Minoru. Realization of a projectively flat space. Tensor (N.S.) **3**, 128–130 (1954).

In this paper the author, using E. Cartan's notations and methods, establishes this theorem: "Let R_n be a space with affine connection without torsion whose holonomy group preserves volume and which is projectively flat but is not flat. Then R_n can be realized as a space of affine connection induced by a hypersurface S_n in an $(n+1)$-dimensional

affine space R_{n+1} and vectors from a fixed point in the space R_{n+1} to the points on the hypersurface." *M. Decuyper.*

Sasayama, Hiroyoshi. On the extended cohomology of higher order. Tensor (N.S.) **3**, 123–127 (1954).

In a previous paper [Tensor (N.S.) **2**, 36–46 (1952); these Rev. **14**, 690] the author considered the extended multiple integral of higher order of the skew-symmetric excovariant extensor field (exfield) in the manifold of higher order and the calculus of the expressions appearing under the integral sign, that is, the extended exterior differential form (exform). The purpose of the present paper is to investigate the relation between the homology of an ordinary point manifold and exfields on the manifold of line-elements of higher order derived from it, and it will be shown that the Betti number of the base space is equal to the maximal number of closed exfields of H. V. Craig's derived extensors, linearly independent in the sense of extended cohomology among these exfields, which is consequently independent of the order of the considered line-elements. (From the author's summary.) *A. Nijenhuis* (Princeton, N. J.).

Raševskiĭ, P. K. Linear differential-geometric objects. Doklady Akad. Nauk SSSR (N.S.) **97**, 609–611 (1954). (Russian)

By differential group D_v of class v is understood the group of linear transformations in a space V_n extended v times. This means that the coefficients are $x_{i_1}^{i'}, x_{i_1 i_2}^{i'}, \cdots, x_{i_1 i_2 \cdots i_v}^{i'}$ computed at some fixed point. This group decomposes into the semidirect product of its semisimple component S, $x_i^{i'}$ $\det|x_i^{i'}|=1$ and its radical $\omega\delta_{i_0}^{i'}, x_{i_1 i_2}^{i'} \cdots x_{i_1 \cdots i_v}^{i'}$. A representation of this group D_v is given by the transformation of a scalar function φ and its partial derivatives up to order v. Denoting the totality of components $i_1, i_1 i_2, \cdots, i_1 \cdots i_v$ arranged in lexicographic order by I, φ_I are then the components of a supervector. A contravariant supervector ψ^I is obtained by requiring $\varphi_I \psi^I$ to be a scalar. A supertensor $Z_{K_1 \cdots K_q}^{I_1 \cdots I_p}$ is obtained in the usual manner and its transformations constitute all the representations of class v.
M. S. Knebelman (Pullman, Wash.).

Raševskiĭ, P. K. On linear representations of nonsemisimple Lie groups with nilpotent radical. Doklady Akad. Nauk SSSR (N.S.) **97**, 781–783 (1954). (Russian)

A Lie group G has an algebra E whose semisimple component forms a space S and whose nilpotent radical forms a space R. If a and b are vectors in R, the product is $c^\gamma = C^\gamma{}_{\alpha\beta} a^\alpha b^\beta$, where $C^\gamma{}_{\alpha\beta}$ are the structure constants of the group. The author considers a system of covariant tensors $V, V_{\alpha_1}, V_{\alpha_1\alpha_2}, \cdots, V_{\alpha_1\cdots\alpha_k}$ which he calls an aggregate and which are subject to the condition

$$V_{\alpha_1\cdots[\alpha_s\alpha_{s+1}]\cdots\alpha_k} = V_{\alpha_1\cdots\beta\cdots\alpha_k} C^\beta{}_{\alpha_s\alpha_{s+1}}.$$

A system of such aggregates $V^i{}_{\alpha_1\cdots\alpha_m}$ with $i=1, \cdots, N$, N being the dimensionality of S, which transform line contravariant vectors in S form the representation space of the group. A dual theorem is proved for a space $W_i{}^{\alpha_1\cdots\alpha_k}$.
M. S. Knebelman (Pullman, Wash.).

de Rham, Georges. Sur la division de formes et de courants par une forme linéaire. Comment. Math. Helv. **28**, 346–352 (1954).

It is well-known that if ω is a 1-form on a differentiable manifold, which does not vanish at a point P, then a necessary and sufficient condition that a p-form α can be written as $\alpha=\omega\wedge\beta$ at P is that $\alpha\wedge\omega=0$. This paper extends this result to give a theorem in the large, and also to the case in which α belongs to a restricted class (e.g., α is to be a C^∞ form with compact carrier in R^n) and it is required to write $\alpha=\omega\wedge\beta$, where β belongs to the same restricted class. The case of C^∞ form with compact carriers is used to extend the result to distributions and currents; it is shown that if T is a distribution $T\wedge\omega=0$ is a necessary and sufficient condition that T should be a linear combination of Dirac distributions relative to the zeros of ω (supposed finite in number), and that, if T is a current of degree >0, $T\wedge\omega=0$ is a necessary and sufficient condition that T should be divisible by ω.

An application is made to the theory of distributions invariant for the group of transformations which leave invariant an indefinite quadratic form.
W. V. D. Hodge (Cambridge, England).

Kinoshita, Toichiro. Families of spinor fields. Physical Rev. (2) **96**, 199–201 (1954).

The author gives a method for classifying systems of non-commuting spinor fields into sets called families. These are determined by the form of the interaction term in the Lagrangean from which the equations of motion of the spinor fields are derived. The author applies his method to the classification of observed spinor fields and discusses the structure of families. *A. H. Taub* (Urbana, Ill.).

NUMERICAL AND GRAPHICAL METHODS

***Lösch, Friedrich. Siebenstellige Tafeln der elementaren transzendenten Funktionen.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. viii+335 pp. DM 49.80.

This volume contains the following tables. (a) $\sin x$, $\cos x$, $\tan x$, $\sinh x$, $\cosh x$, $\tanh x$, $\log x$, e^x, e^{-x}, arc sin x $(x<1)$, arc tan x, arg sinh x, arg tanh x $(x<1)$, arg cosh x $(x>1)$, arg coth x $(x>1)$, for $x=0(0.0001)0.1$ (9D);

$$x=0.1(0.0005)3.15, \quad 3(0.01)10(0.1)20 \text{ (7D)};$$

with first differences when $x<3.15$, together with $\tan x$, $x=1.5680(0.0001)1.5730$, arg tanh x, $x=0.9980(0.0001)1$ and arg coth x, $x=1(0.0001)1.0020$, all to 7D. (b) $\sin x$, $\cos x$, $\log x$, e^x, e^{-x}, $\sinh^{-1} x$, $\cosh^{-1} x$ for $x=0(1)100$ (7D, except for e^x, e^{-x} which are 7S). (c) $\frac{1}{2}\pi n$, $n=0(1)100$ (12D). (d) $\sin \frac{1}{2}\pi x$, $\cos \frac{1}{2}\pi x$ for $x=0(0.001)0.5$ (7D). (e) $e^{\pi x/2}$, $e^{-\pi x/2}$, $\sinh \frac{1}{2}\pi x$, $\cosh \frac{1}{2}\pi x$ for $x=0(0.01)2$ (7D). (f) $e^{\pi x/180}$, $e^{-\pi x/180}$, $\sinh(\pi x/180)$, $\cosh(\pi x/180)$ for $x=0(1)180$ (7D). (g) Conversion tables from degrees to radians and vice versa. (h) A short table of miscellaneous constants.
J. A. Todd (Cambridge, England).

Have

***Tables of functions and of zeros of functions. Collected short tables of the National Bureau of Standards Computation Laboratory.** National Bureau of Standards Applied Mathematics Series, no. 37. U. S. Government Printing Office, Washington, D. C., 1954. ix+211 pp. $2.25.

A collection of short tables, all but two of which have been published elsewhere and most of which have been reviewed in Mathematical Reviews. The two new tables are a table of $\sin x$ and $\cos x$ for $x=100(1)1000$, 8D, and a table of $x^n/n!$ for $x=.01(.01)1.99$, 13D, n being taken for each x up

to the point where $x^n/n!$ vanishes in the last decimal place, and also for $x=1(1)10$, $n=1(1)40$, 8 significant figures. The tables are reproduced by photo-offset from the originals.

Have

***Fox, L. A short table for Bessel functions of integer orders and large arguments.** Royal Society Shorter Mathematical Tables, No. 3. Cambridge, at the University Press, 1954. 28 pp. 6s.6d. \$1.25.

Previous volumes [British Association for the Advancement of Science, Math. Tables, v. 6, 10, Cambridge, 1937, 1952; these Rev. 14, 410] have given the values of the Bessel functions $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$ for integral values of n from 0 to 20, and for x ranging to 20 or 25. The aim of the present work is to extend these tables to cover the range $20\leqq x\leqq\infty$, for the same values of n.

For this purpose the argument $1/x$ was chosen for all four functions, and the asymptotic expansions

$$J_n(x)=(2/\pi x)^{1/2}[P_n(x)\cos\theta-Q_n(x)\sin\theta],$$
$$Y_n(x)=(2/\pi x)^{1/2}[P_n(x)\sin\theta+Q_n(x)\cos\theta],\ \theta\geqq x-\tfrac{1}{2}n\pi-\tfrac{1}{4}\pi,$$
$$I_n(x)=(2\pi x)^{-1/2}e^xF_n(x),\quad K_n(x)=(\pi/2x)^{1/2}e^{-x}G_n(x),$$

introducing auxiliary functions of n and x: P, Q, F, G.

The tables contain values of $P_n(x)$, $Q_n(x)$, $F_n(x)$, $G_n(x)$ for $1/x=[0(.001).05;\ 9D]$ for $n<10$; and of $P_n(x)$, $Q_n(x)$, $\ln F_n(x)$, $\ln G_n(x)$ to 8D for $n\geqq 10$; modified second differences and fourth differences, where significant, are printed for use in interpolation. With the aid of tables of sines and cosines, such as "Tables of circular and hyperbolic sines and cosines for radian arguments" [Math. Tables Project, Nat. Bur. Standards, New York, 1939], J_s and Y_s are readily found.

Details of the computation of P, Q, F, G, are given in the five pages of the Introduction by L. Fox, who, on behalf of the Mathematical Tables Committee planned the computation of the tables. These were, however, computed by a group at the National Physical Laboratory. The 21 pages of the tables were printed by photo-lithography from typed sheets. *R. C. Archibald* (Providence, R. I.).

Have

***Table of the gamma function for complex arguments.** National Bureau of Standards Applied Mathematics Series, No. 34. U. S. Government Printing Office, Washington, D. C., 1954. xvi+105 pp. \$2.00.

These excellent and long-needed tables give 12 decimal values of U and V, where $U+iV=\ln\Gamma(x+iy)$ for both $x, y=0(0.1)10$. The table for each value of x occupies one page, so that, for example, the important tables of $\Gamma(iy)$ and $\Gamma(\frac{1}{2}+iy)$ appear on pages 2 and 52 respectively. Auxiliary tables give 15-decimal values of $\sin\pi x$ and $\cos\pi x$, and 15-figure values of $\sinh\pi x$ and $\cosh\pi x$, also for $x=0(0.1)10$.

A useful introduction gives some properties of the Gamma function and considerable details of methods of interpolation. There is also a description of the preparation of the tables and a bibliography of earlier tables, helpful auxiliary tables and some other references. *J. C. P. Miller.*

Sharp, W. T., Kennedy, J. M., Sears, B. J., and Hoyle, M. G. Tables of coefficients for angular distribution analysis. Atomic Energy of Canada Ltd., Chalk River, Ont., Rep. **CRT-556**, ii+xxxix+38 pp. (1953).

"Tables of coefficients for angular distribution analysis are presented in factored form for a range of parameters appropriate to the needs of the Chalk River electrostatic oscillator. In particular these tables suffice for most double correlation experiments involving particles with orbital angular momentum not larger than 4 or dipole, quadruple, or octupole gamma radiation. Some of the coefficients are taken from previous tabulations while the remainder have been computed." (From the author's abstract.)
A. Erdélyi (Pasadena, Calif.).

Have

***Table of salvo kill probabilities for square targets.** National Bureau of Standards Applied Mathematics Series, no. 44. U. S. Government Printing Office, Washington, D. C., 1954. ix+33 pp. 30 cents.

Let

$$P_R(\xi,\eta)=\frac{1}{\pi\sigma_R^2}\int_{-a}^{a}\int_{-a}^{a}\exp\left[-\frac{(x-\xi)^2}{\sigma_R^2}-\frac{(y-\eta)^2}{\sigma_R^2}\right]dxdy,$$
$$Q(\xi,\eta)=1-[1-P_KP_R(\xi,\eta)]^N,$$
$$P_{SK}=\frac{1}{\pi\sigma_A^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q(\xi,\eta)\exp\left[-\frac{\xi^2}{\sigma_A^2}-\frac{(\eta-y_0)^2}{\sigma_A^2}\right]d\xi d\eta.$$

Tables are presented of $P_{SK}(P_K, y_0, \sigma_A, \sigma_R, N)$ for $P_K=.1$, $.4$, $.7$, 1.0; $y_0=0$, a, $2a$, $4a$, $7a$, $11a$, $16a$, $22a$; σ_R, $\sigma_A=a$, $2a$, $4a$, $11a$, $16a$, $22a$; $N=1$, 5, 10, 25, 50, 100, 150, 200; 4D.

Morduchow, Morris. Method of averages and its comparison with the method of least squares. J. Appl. Phys. 25, 1260–1263 (1954).

The author compares the method of averages with the method of least squares for determining the coefficients in an assumed functional form to secure an approximation to a given curve. The analysis is carried out in detail for the case of fitting a straight line to a set of n points (ξ_i, y_i), where the ξ_i are equally spaced. A simple change of variables ξ to x leads to points (x_i, y_i), where $x_i=i$. If $y=Ax+B$ is the result obtained by the method of averages, and $y=\alpha x+\beta$ is the result obtained by the method of least squares, it is shown that

$$\alpha=A+\frac{12M}{n(n^2-1)},\quad \beta=B-\frac{6M}{n(n+1)},$$

where M is essentially the first-order moment of the residuals from the line obtained by the method of averages. From these formulas a number of interesting and useful conclusions are drawn, including an estimate of the maximum discrepancy between the results of the two methods.
W. E. Milne (Corvallis, Ore.).

Frame, J. S. Some trigonometric, hyperbolic and elliptic approximations. Amer. Math. Monthly 61, 623–626 (1954).

Certain close approximations for trigonometric and hyperbolic functions and for incomplete elliptic integrals of the first and second kinds are obtained. For example, the fourth-order approximation $(2+\cos\theta)/3$ for $\sin\theta/\theta$ is made stronger by a certain type of correction for the error.
P. W. Ketchum (Urbana, Ill.).

Romberg, W., und Viervoll, H. Darstellung eines Kurvenstückes durch wenige Exponentialfunktionen (Differentialgleichungsmethode). Arch. Math. Naturvid. 52, 57–63 (1954).

The authors are interested in the expansion of functions $f(t)$ in series of the form $\sum_1^N c_k\exp(z_kt)$, where f is assumed known at a finite number of points on the interval $(0, T)$. They show how the c_k and the z_k can be determined and also how to find the minimal number $n\leqq N$ of c_k which are nonzero. The method of analysis is based upon inner products of the successive derivatives of f and of a set φ_k $(k=0, 1, \cdots, N)$ of linearly independent functions. *H. H. Goldstine.*

★Šafarevič, I. R. O rešenii uravnenii vysših stepeneĭ. [**On the solution of equations of higher degrees.**] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 24 pp. .40 rubles.

No. 15 in the series, Popular lectures on mathematics.

Millington, G. A note on the solution of the sextic equation. Quart. J. Mech. Appl. Math. 7, 357–366 (1954).

An iterative process is given for the solution of a sextic equation with real coefficients. The method starts with the factorization of the sextic polynomial into two cubic factors. The process requires an initial choice of only one quantity, but at the final stage requires the solution of a cubic equation that may have complex coefficients. The method can be made rapidly convergent and the author finds it compares favorably with others in speed and ease of handling.

E. Frank (Chicago, Ill.).

Morris, J., and Head, J. W. A note on the escalator process. Routh's stability criteria for polynomial characteristic equations derived by algebra. Aircraft Engrg. 26, 388–389 (1954).

Here are derived Routh's criteria for a Hurwitz polynomial of even degree, in particular for a sextic polynomial. The case of damped Lagrangian frequency equations is included for practical interest. *E. Frank.*

Kuntzmann, Jean. Etude de représentations approchées de dérivées. C. R. Acad. Sci. Paris 239, 1110–1111 (1954).

Conditions are found in order that a Lagrange-type interpolation formula for a derivative of order q be definite. By definite is meant that the error of an $(n+1)$-point formula, when expressed in the form of a weighted integral over all real t of $f^{(n+1)}(t)$, shall have a weight function $K(t)$ which does not change sign. [Cf. W. E. Milne, Numerical calculus, Princeton, 1949, p. 115; these Rev. 10, 483.] The conditions are expressed in terms of a modified Vandermonde determinant formed on the points of interpolation, with the qth powers omitted. *P. W. Ketchum* (Urbana, Ill.).

Davis, P., and Rabinowitz, P. On the estimation of quadrature errors for analytic functions. Math. Tables and Other Aids to Computation 8, 193–203 (1954).

Error estimates are obtained for the class of functions $\{f\}$ analytic inside and on an ellipse whose major axis contains the interval of integration. For a fixed quadrature formula the error estimate is obtained as the product of the L^2 norm of f taken over the ellipse times a constant σ which depends on the geometry of the ellipse. A table of values of σ is given for various quadrature formulas and ellipses. In applications the norm of f must be estimated and then the ellipse is found which minimizes the error. Examples are given in which the results are superior to error estimates based on higher derivatives of f. *P. W. Ketchum.*

Shue, G. L. Simplified numerical integration. A successive approximation difference-table method for the solution of differential equations. Aircraft Engrg. 26, 89–94, 96 (1954).

This is an elementary exposition of the numerical integration of ordinary differential equations by means of finite differences. The explanations are given in minute detail and enough examples are worked so that a beginner can easily understand the procedure. *W. E. Milne.*

Laville, G. Résolution graphique d'intégrales utilisées en analyse harmonique et en calcul symbolique. Ann. Fac. Sci. Univ. Toulouse (4) 16 (1952), 153–168 (1 plate) (1953).

The integral of $f(\theta)e^{i\theta}$ between arbitrary limits is found for the case where $f(\theta)$ is a polynomial by means of a rectangular polygonal construction resembling the method of Lill [cf. Fr. A. Willers, Practical analysis, Dover, New York, 1947, p. 79; these Rev. 10, 404] for the evaluation of polynomials. For a general $f(\theta)$ this construction is applied to polynomial approximations to $f(\theta)$ in successive subdivisions of the range of integration. The method is used to calculate complex Fourier coefficients and integrals. A detailed example is included. *P. W. Ketchum.*

★Fox, L. Practical solution of linear equations and inversion of matrices. Contributions to the solution of systems of linear equations and the determination of eigenvalues, pp. 1–54. National Bureau of Standards Applied Mathematics Series No. 39. U. S. Government Printing Office, Washington, D. C., 1954. $2.00.

Among the many known direct methods for solving linear algebraic systems, most can be separated into three classes: triangularization, orthogonalization, and modification. Within these classes the differences are mainly in the details of the computational layout, or in the point of view. Methods which fall into the first class are the method of elimination, and methods due to Doolittle, Choleski (square-rooting), Crout, Banachiewicz, Aitken, and the present author. In Jordan's method the triangular form does not appear explicitly because the back substitution accompanies the triangularization. The usual methods of the second class orthogonalize the unit vectors, or else the column vectors or the row vectors of the matrix itself. The method of conjugate gradients selects the vectors to be orthogonalized as the computation proceeds. Methods of the third class, which include partitioning, obtain the inverse of a matrix which differs in certain rows or columns or both from a matrix whose inverse is already known.

Of the indirect methods, the most common, as applied to the system $Ax=h$, use an iteration $A_1x_p=h-A_2x_{p-1}$, where $A=A_1+A_2$. In one method A_1 is a diagonal matrix whose diagonal elements are those of A. In another (Gauss-Seidel), A_1 is triangular and agrees with A along and below the diagonal. A modification of the latter method replaces the system of order n by one of order $n+1$ requiring iteration by a matrix T, of which some power can be used for accelerating convergence.

The present paper gives computational layouts, and brief but entirely lucid and elementary expositions and comparisons of the method of partitioning but not the general principle of modification, and of all the methods mentioned in the other classes except for the method of conjugate gradients. *A. S. Householder* (Oak Ridge, Tenn.).

Rahman, A. Numerical evaluation of determinants. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 40, 798–801 (1954).

For the evaluation of numerical determinants by elimination with desk calculators, the author advocates use of submatrices of orders 2 or 3 as "pivots". This corresponds in the solution of linear systems to the simultaneous elimination of several unknowns, i.e., to "block elimination". The author claims a certain saving of effort by his procedure, but offers no argument to justify this claim. There is a numerical example of order 8. *G. E. Forsythe.*

Collatz, L. Zur Fehlerabschätzung bei linearen Gleichungssystemen. Z. Angew. Math. Mech. **34**, 71–72 (1954).

Let x exactly solve the finite linear system $Ax=r$. Because of round-off or other errors, suppose one calculates instead with $A+F$, $r+s$, and gets an approximate solution y for which $(A+F)y=r+s+f$. It is shown that

$$(*)\qquad |y-x|\leqq\frac{|s|+|f|+|\bar{E}|\,|y|}{|\underline{A+F}|-|\bar{F}|},$$

where $|z|^2=\sum z_i^2$, and where $|\underline{G}|$, $|\bar{G}|$ are the lower, upper bounds of G in the euclidean metric. Now the upper bound in (*) is easily estimated, but the lower bound of $A+F$ is ordinarily not accessible. When $A+F$ is positive definite hermitian, however, estimates are known. There is a numerical example in which the author uses the third estimate of Bartsch for the lower bound [see the following review].

G. E. Forsythe (Los Angeles, Calif.).

Bartsch, Helmut. Abschätzungen für die kleinste charakteristische Zahl einer positiv-definiten hermiteschen Matrix. Z. Angew. Math. Mech. **34**, 72–74 (1954).

$A=(a_{ij})$ is of order n, with conjugate transpose $\bar{A}'$. Let $B=\bar{A}'A$; $s_A=\operatorname{trace}A$; $s_B=\operatorname{trace}B$; $a=\det A$. Let ρ_A be the Rayleigh quotient of A corresponding to a fixed vector $x=(x_i)$:

$$\rho_A=\sum_{i,j=1}^{n}\bar{x}_ia_{ij}x_j\Big/\sum_{i=1}^{n}\bar{x}_ix_i.$$

Define ρ_B analogously. Let $\kappa_1,\cdots,\kappa_n$ be the characteristic numbers of A.

If A is positive-definite hermitian, define $\kappa_{\min}=\min_i\kappa_i$. Eight lower bounds for $\kappa_{\min}$ are proved and compared, of which the first two are due to Wittmeyer [Dissertation, Darmstadt, 1934], and several follow ideas of Collatz. There is a numerical example and a statement of how to apply the inequalities to nonhermitian matrices.

The eight lower bounds for $\kappa_{\min}$ are:

$$a[(n-1)s_B^{-1}]^{(n-1)/2};\quad a(n-1)^{n-1}s_A^{1-n};$$
$$a(n-1)(n-2)^{n-2}s_A^{n-3}(s_A^2-s_B)^{2-n};$$
$$a\rho_B^{-1/2}[(n-2)(s_B-\rho_B)^{-1}]^{(n-2)/2}$$

provided $\rho_B(n-1)\geqq s_B$; $a\rho_A^{-1}(n-2)^{n-2}(s_A-\rho_A)^{2-n}$ provided $\rho_A(n-1)\geqq s_A$; the least positive root κ of

$$\kappa^2=a^2(n-1)^{n-1}(s_B-\kappa^2)^{1-n};$$

the least positive root of $\kappa=a(n-1)^{n-1}(s_A-\kappa)^{1-n}$.

G. E. Forsythe (Los Angeles, Calif.).

Stesin, I. M. Computation of eigenvalues by means of continued fractions. Uspehi Matem. Nauk (N.S.) **9**, no. 2(60), 191–198 (1954). (Russian)

Let A be a symmetric matrix or linear operator, and let $c_k=(A^kb,b)$ or $f(A^kb)$, where f is an arbitrary linear functional. L. A. Lyusternik [in an unavailable report of 1952] proposed a method of getting the eigenvalues of A by converting the series (I) $\sum_{k=0}^{\infty}c_kz^{-k-1}$ into a continued fraction of type

$$(\mathrm{II})\qquad \frac{k_0\,|}{|z+l_0}+\frac{k_1\,|}{|z+l_1}+\cdots+\frac{k_n\,|}{|z+l_n}+\cdots.$$

The present author develops an algorithm for converting (I) into a continued fraction of type

$$(\mathrm{III})\qquad \frac{1\,|}{|b_1z}+\frac{1\,|}{|b_2z}+\frac{1\,|}{|b_3z}+\cdots,$$

whose nth convergent is $P_n(z)/Q_n(z)$. For an integral equation with a symmetric kernel $K(x,y)$ he shows that the zeros of $Q_n(-z)$ converge to the reciprocals κ_i of the eigenvalues λ_i of $K(x,y)$. The method is closely related to that of Lanczos [J. Research Nat. Bur. Standards **45**, 255–282 (1950); these Rev. **13**, 163].

Reviewer's remarks: The author does not seem to have carried the algorithm as far as Rutishauser [Z. Angew. Math. Physik **5**, 233–251 (1954); these Rev. **16**, 176], whose QD-algorithm yields the λ_i as limits of numbers in the algorithm, without having to find $Q_n(-z)$ or its zeros. An example would help the reader follow Stesin's work.

G. E. Forsythe (Los Angeles, Calif.).

Gouarné, René. Remarques sur la méthode des polygones. C. R. Acad. Sci. Paris **239**, 383–385 (1954).

The chemist Samuel [same C. R. **229**, 1236–1237 (1949)] introduced a "method of polygons" for computing the characteristic polynomial of a matrix with few non-zero elements. Gouarné explains the method of polygons in terms of the structure of the symmetric group of permutations on n marks, and relates it to the partitions of the integer n. He also expresses the characteristic polynomial of A as a function of the traces of the powers of A. [The citation C. R. Acad. Sci. Paris **237**, 237 (1953) for an earlier paper by Gouarné must be in error.] *G. E. Forsythe.*

Thomas, L. H. Computation of one-dimensional compressible flows including shocks. Comm. Pure Appl. Math. **7**, 195–206 (1954).

Two generalizations of Adams' method for numerical integration of ordinary differential equations are described in which the increment of the independent variable can be varied from step to step. These are applied to devise an efficient method for solution of canonical hyperbolic systems of partial differential equations, in particular those of one-dimensional unsteady flow referred to characteristic variables. The crux of the method is the following. It is not justified to take differences across a shock or a characteristic at which some partial derivative of some flow function is discontinuous. To overcome this difficulty, small intervals and first backward differences of derivatives are used to start a computation from a line of discontinuity or any initial curve. Thereafter, with increasing distance from the line of discontinuity the interval size and order of backward differences can be increased gradually to convenient values selected in accordance with a well known criterion for minimizing the truncation error in Adams' method.

J. H. Giese (Havre de Grace, Md.).

Lotkin, Mark. The propagation of error in numerical integrations. Ballistic Research Laboratories, Aberdeen Proving Ground, Md. Rep. No. **875**, 30 pp. (1953).

The propagation of error in the numerical integration of $y^{(n)}=f_n(x,y,y',\cdots,y^{(n-1)})$, subject to suitable initial conditions is studied by use of the variational difference equations [see the preceding review and H. Rutishauser, Z. Angew. Math. Physik **3**, 65–74 (1952); these Rev. **13**, 692]. The general results obtained are specialized algebraically to deal with such familiar methods as those of Heun, Euler, Simpson. A detailed arithmetic account of the integration by Simpson's rule of $y'=y-2x/y$ is given; the estimates attained are seen to be quite realistic. *J. Todd.*

Maehly, Hans J. Ein neues Variationsverfahren zur genäherten Berechnung der Eigenwerte hermitescher Operatoren. Helvetica Phys. Acta 25, 547–568 (1952).

The method mentioned in the title gives upper and lower bounds for eigenvalues, provided "the positions of the neighboring eigenvalues can be roughly estimated". As the author states, it does not differ essentially from the method proposed for integral equations by N. J. Lehmann [Z. Angew. Math. Mech. 29, 341–356 (1949); 30, 1–16 (1950); these Rev. 11, 599]. The intuitive idea behind it is roughly the following: if p is not an eigenvalue of the Hermitian operator A, and is also not a limit of eigenvalues of A, then the operator $(A-pI)^{-1}$ will be bounded, the method of Ritz will be readily applicable to it, and this will furnish bounds on the eigenvalues of A. The method in question is applied to the computation of the two lowest eigenvalues of the following problem:

$$\frac{d^2\Phi}{dx^2}-\kappa^2\Phi+\lambda\frac{e^{-x}}{x}\Phi=0,\quad 0\leqq x<\infty,$$

$$\Phi(0)=0,\quad \int_0^\infty \varphi^2(x)\frac{e^{-x}}{x}dx<\infty,$$

which occurs in the deuteron-problem [for the literature see L. Hulthén and K. V. Laurikainen, Rev. Modern Physics 23, 1–9 (1951); these Rev. 12, 862]. *J. B. Diaz.*

Ku, Y. H. Nonlinear analysis of electro-mechanical problems. J. Franklin Inst. 255, 9–31 (1953).

The author considers the initial-value problem of the ordinary 2nd order nonlinear equation

$$\ddot{x}+f(\dot{x},x)+f_1(x)=F(t) \tag{1}$$

and, in particular, the case $F(t)=0$. Introducing $v=\dot{x}$, (1) may be reduced to the system of 1st order equations

$$\frac{dv}{dx}=\frac{-f(v,x)-f_1(x)+F(t)}{v} \tag{2a}$$

$$\frac{dt}{dx}=\frac{1}{v} \tag{2b}$$

which is solved by graphical step-by-step methods. The introduction of x as independent variable is relevant in the particular case $F(t)=0$ when (2b) is omitted. Worked examples include illustrations on the van der Pol equation

$$\ddot{x}-k(1-x^2)\dot{x}+x=F(t).$$

No rigorous study of the accuracy of the graphical step-by-step procedure is offered nor is the accumulation of rounding errors investigated. *H. O. Hartley* (Ames, Iowa).

Mysovskih, I. P. Application of Čaplygin's method to the solution of the Dirichlet problem for a special type of elliptic differential equations. Doklady Akad. Nauk SSSR (N.S.) 99, 13–15 (1954). (Russian)

Consider the equation $\Delta u=f(x,y,u)$ in a bounded, simply-connected region D with sufficiently smooth boundary S. Suppose u vanishes on S. If $f_u>0$ and $f_{uu}\leqq 0$ in and on the boundary of D, the author defines the sequence of functions u_n with $u_0=0$ satisfying

$$\Delta(u_n-u_{n-1})=f_u(x,y,u_{n-1})(u_n-u_{n-1})+f(x,y,u_{n-1})-\Delta u_{n-1},$$

and shows that $u_n\leqq u_{n-1}$ and that $u_n\to u$. Moreover, if $f(x,y,V)-\Delta V\leqq 0$, and V vanishes on S, then $u_n\geqq V$.

A sequence of functions v_n with $v_n\geqq v_{n-1}$ is defined by means of a somewhat more complicated recursion where also $v_n\to u$. The case $f_u\geqq 0$ with no assumption on f_{uu} is also treated, but again the recursions are more complicated.

A. S. Householder (Oak Ridge, Tenn.).

Horvay, G., and Spiess, F. N. Orthogonal edge polynomials in the solution of boundary value problems. Quart. Appl. Math. 12, 57–69 (1954).

The authors give numerical comparisons between exact solutions to certain classical boundary-value problems with elementary boundaries (e.g., triangles, rectangles, circular sectors) and approximate solutions, the latter obtained by changing the corresponding variational problem into a more elementary one, not precisely equivalent: if $\Phi(x,y)$ is the desired solution, and the boundary values are Φ_b over one portion, say Γ, of the boundary, zero elsewhere, then the approximation is

$$\Phi\sim\sum c_n f_n g_n.$$

Here $\{f_n\}$ is a set of functions which are polynomials in the boundary parameter along Γ, vanish on the remaining boundary, and are orthonormal with respect to a scalar product on Γ appropriate to the form of the Dirichlet integral. The c_n are Fourier coefficients of Φ_b with respect to the f_n, and g_n are functions of the variable held constant along Γ. These are found for each n by an Euler-Lagrange condition to minimize the Dirichlet integral of $\sum A_n f_n g_n$, assuming that $\{f_n g_n\}$ are orthogonal to each other in the double integration. [Cf. H. Poritsky, Proc. 5th Internat. Congress Appl. Mech., Cambridge, Mass., 1938, Wiley, New York, 1939, pp. 700–707; Kantorovitch, Izvestiya Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) 1933, 647–652.]

Second and higher-order approximation methods are indicated by altering the last assumption. Questions of convergence to the exact solution are not discussed, nor is the ease of application compared with other techniques in case of complicated boundaries. More general boundary values are handled by superposition, suggesting a prohibitive amount of labor even for first-order approximation in the case of many-sided polygons, for example. The sample calculations indicate the approximation of eigen-values as well as eigen-functions. *A. B. Novikoff.*

Burgerhout, Th. J. On the numerical solution of partial differential equations of the elliptic type. I. Appl. Sci. Research B. 4, 161–172 (1954).

The coefficient matrix associated with the finite-difference equations for the potential equation has a special form (a "chain" matrix with elements that are "chain" matrices). The explicit determination of the inverse matrix and the development of efficient relaxation methods are described.

E. Isaacson (New York, N. Y.).

Schröder, Johann. Zur Lösung von Potentialaufgaben mit Hilfe des Differenzenverfahrens. Z. Angew. Math. Mech. 34, 241–253 (1954). (English, French and Russian summaries)

The author observes that the unknowns which appear in the finite-difference equations arising from potential theory fall into two natural sets. Advantage is taken of this fact to obtain more quickly convergent iterative procedures for the solution of the linear equations. Application is made to a boundary-value problem and to a characteristic-value problem. *E. Isaacson* (New York, N. Y.).

Dahlquist, Germund. Convergence and stability for a hyperbolic difference equation with analytic initial-values. Math. Scand. 2, 91–102 (1954).

The author considers the second-order differential problem

$$D_{tt}F(x,t)=D_{xx}F(x,t),\quad F(x,0)=f(x),\quad D_tF(x,0)=g(x),$$

and the corresponding difference problem

$$\delta_{tt}\phi(x,t)=\delta_{xx}\phi(x,t),\quad \phi(x,0)=f(x),\quad \Delta_t\phi(x,0)=g(x),$$

where D_{xx}, D_{tt}, D_t are partial differential operators and δ_{xx} δ_{tt}, Δ_t are partial divided-difference operators; δ indicates central differences and Δ forward differences. x is a continuous variable in the closed interval I. Does $\phi(x,t)$ converge to $F(x,t)$ as Δx, Δt approach zero, with constant mesh-ratio $u=\Delta x/\Delta t$? Courant, Friedrichs and Lewy [Math. Ann. **100**, 32–74 (1928)] proved that for $u<1$ convergence is impossible for general f, g. The author points out that, for f, g analytic in I, convergence can occur when $u<1$, and discusses the extent of the regions of convergence. He notes however that, when solving the difference problem in the usual recursive numerical way, one cannot use $u<1$, since then the unavoidable round-off errors introduce non-analyticities which grow rapidly and invalidate the calculations beyond a certain point ("instability"). The paper concludes with a brief discussion of convergence in the differential-difference case: $\Delta x=0$, $\Delta t\to 0$.

M. A. Hyman (Pittsburgh, Pa.).

Eliasen, Erik. Numerical solutions of the perturbation equation for linear flow. Tellus 6, 183–191 (1954).

The author considers certain two-dimensional linear flows and by a numerical procedure obtains some indications as to the conditions for their instability. He further, by means of some approximations, obtains expansions for the flows that are valid near the beginnings of the motions. The plausibility of these heuristic techniques is indicated by a consideration of the Couette flow. The paper closes with a discussion of a symmetrical harmonic velocity function as it relates to the instability of a barotropic atmosphere.

H. H. Goldstine (Princeton, N. J.).

Booth, Andrew D. Reciprocals—a note on a computer method for finding them. Computers and Automation 3, no. 7, 16, 25 (1954).

Steinhaus, H. A dispersiometer. Zastosowania Mat. 1, 321–329 (1954). (Polish. Russian and English summaries)

A "dispersiometer" makes it possible to calculate such expressions as

$$\bar{s}=\lambda_n[\xi_1^2+\cdots+\xi_n^2]^{1/2}=\lambda_n[(x_1-\bar{x})^2+\cdots+(x_n-\bar{x})^2]^{1/2},$$

n being constant and not greater than 20, merely by taking readings of ordinary scales, sticking pins and sliding a plate upon a drawing board. *From the author's summary.*

Erismann, Theodor. Theorie und Anwendungen des echten Kugelgetriebes. Z. Angew. Math. Physik 5, 355–388 (1954).

The brief treatment given to ball integrators (Henrici-Coradi and Amsler) in textbooks is here elaborated in the manner of Hele Shaw [Philos. Trans. Roy. Soc. London **176**, 367–402 (1885)]. The ball is driven by a short cylindrical roller, and in turn drives one or more similar cylindrical rollers with fixed or movable axes. Various combinations of two or more balls are also discussed. They can be used as multipliers, component solvers, integrators, differentiators and frequency analyzers. Photographs of available commercial applications are shown in planimeters, speedometers, dynamometers and variable drives.

M. Goldberg (Washington, D. C.).

Glossary of terms in the field of computers and automation. Computers and Automation 3, no. 10, 8–23 (1954).

Revesz, G. An autocorrelogram computer. J. Sci. Instruments 31, 406–410 (1954).

Leutert, W. W. Simulation of a battle on high speed digital computers. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. No. **911**, ii+48 pp. (1954).

Campbell, Lloyd W. An ORDVAC floating binary code checker. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. No. **823**, 30 pp. (1954).

Nikolaev, P. V. Binary anamorphosis of functions. Doklady Akad. Nauk SSSR (N.S.) 88, 209–212 (1953). (Russian)

This paper extends the concepts and uses the methods of the author's earlier work [especially same Doklady (N.S.) **67**, 421–423; **68**, 229–232 (1949); these Rev. **11**, 406] to study the possibility of representing $F(t,\tau)=F(t_1,\tau_1;\cdots;t_n,\tau_n)$ as a binary Massau determinant $|f_{1i}(t_i,\tau_i);\cdots;f_{in}(t_i,\tau_i)|$. The conditions obtained are analogous to those previously given. Any two anamorphoses of $F(t,\tau)$ with the same pairing of variables are projective if the dimension with respect to at least one pair of variables is 4 for $n=4$, or 3 for $n=3$, and in the case $n=4$ if all other dimensions are at least 3. For $n=3$ and nomographic orders 3 and 4, by proper choice of the elements of the basic resolution of $F(t,\tau)$, it can be reduced to one of five canonical forms, analogous to those of Soreau, Cauchy and Clark. They correspond to the sign of the determinant of the anamorphosis matrix.

R. Church (Monterey, Calif.).

RELATIVITY

Sáenz, A. W. Elementare Herleitung der von Hlavatý angegebenen kanonischen Formen für den elektromagnetischen Tensor in der einheitlichen Feldtheorie. Z. Physik 138, 489–498 (1954).

Let $h_{\lambda\mu}$, $k_{\lambda\mu}$ be the symmetric and skew-symmetric parts of the asymmetric fundamental tensor $g_{\lambda\mu}$ of the unified field-theory of Einstein, $h_{\lambda\mu}$ having signature $(---+)$. Then it is possible to find four real vectors $U_\lambda{}^A$ ($A=$I, II, III, IV) such that $h_{\lambda\mu}$ has the form

$$h_{\lambda\mu}=-U_\lambda{}^{\mathrm{I}}U_\mu{}^{\mathrm{I}}-U_\lambda{}^{\mathrm{II}}U_\mu{}^{\mathrm{II}}-U_\lambda{}^{\mathrm{III}}U_\mu{}^{\mathrm{III}}+U_\lambda{}^{\mathrm{IV}}U_\mu{}^{\mathrm{IV}},$$

and $k_{\lambda\mu}$ has one of the following four mutually exclusive canonical forms:

$$k_{\lambda\mu}=2\alpha U^{\mathrm{I}}_{[\lambda}U^{\mathrm{II}}_{\mu]}+2\beta U^{\mathrm{III}}_{[\lambda}U^{\mathrm{IV}}_{\mu]},$$
$$k_{\lambda\mu}=2\alpha U^{\mathrm{I}}_{[\lambda}U^{\mathrm{II}}_{\mu]},\quad k_{\lambda\mu}=2\beta U^{\mathrm{III}}_{[\lambda}U^{\mathrm{IV}}_{\mu]},$$
$$k_{\lambda\mu}=2\gamma U^{\mathrm{II}}_{[\lambda}U^{\mathrm{III}}_{\mu]}\pm 2\gamma U^{\mathrm{II}}_{[\lambda}U^{\mathrm{IV}}_{\mu]}.$$

This theorem was established by Hlavatý [J. Rational Mech. Anal. **1**, 539–562 (1952); these Rev. **14**, 416] by methods depending upon line geometry, and the present author now establishes the same result by analytical methods. *H. S. Ruse* (Leeds).

Hlavatý, Václav. Maxwell's field in the Einstein unified field theory. Nieuw Arch. Wiskunde (3) **2**, 103–114 (1954).

This paper is not one of the author's series on the Einstein theory [see these Rev. **15**, 654]; it forms, however, a useful adjunct to the series. Its main results are roughly summarized as follows. (i) The unified theory admits exactly one skew-symmetric tensor function $m_{\lambda\mu}$ of the fundamental tensor $g_{\lambda\mu}$ that satisfies the Maxwellian equations $\partial_{[\omega}m_{\mu\lambda]}=0$. This tensor, defined up to a constant factor, is in fact

$$m_{\lambda\mu}=\tfrac{1}{2}\gamma e_{\lambda\mu\alpha\beta}{}^*g^{[\alpha\beta]}\sqrt{|\mathfrak{g}|},$$

where $\mathfrak{g}=\det(g_{\lambda\mu})$, $\gamma=\operatorname{sgn}\mathfrak{g}$, $e_{\lambda\mu\alpha\beta}$ is the fundamental alternating tensor-density of components $(\pm1, 0)$, and ${}^*g^{\lambda\mu}$ is the contravariant tensor defined by ${}^*g^{\lambda\nu}g_{\lambda\mu}=\delta_\mu{}^\nu$. (ii) Infinitely many tensors $m_{\lambda\mu}$ can be found whose pth approximation satisfies the Maxwellian equations for $p\geqq 0$.

The full statement of the theorems contains the phrases "Maxwell field" and "almost Maxwellian field", which are precisely defined by the author. *H. S. Ruse* (Leeds).

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. C_3. Applications. II. J. Rational Mech. Anal. **3**, 645–689 (1954).

[For earlier papers in this series, and related papers, see Hlavatý and/or Sáenz, these Rev. **14**, 416, 505, 1132; **15**, 654; and the two preceding reviews.] Notation: Fundamental tensor $g_{\lambda\mu}$, symmetric part $h_{\lambda\mu}$, skew-symmetric part $k_{\lambda\mu}$; connection coefficients $\Gamma^\nu{}_{\lambda\mu}$; torsion tensor $S_{\lambda\mu}{}^\nu$; Christoffel symbols $\{{}^{\nu}_{\lambda\mu}\}$, formed from $h_{\lambda\mu}$.

In one of the earlier papers [C_2 of the present series, see these Rev. **15**, 654] the author considered the consequence of assuming the tensor $U^\nu{}_{\lambda\mu}$ defined by

$$\Gamma^\nu{}_{\lambda\mu}=\{{}^{\nu}_{\lambda\mu}\}+S_{\lambda\mu}{}^\nu+U^\nu{}_{\lambda\mu}$$

to be zero. Assuming $U^\nu{}_{\lambda\mu}$ to be identically zero places a structural condition upon the manifold, but, inasmuch as it is approximately zero when $k_{\lambda\mu}$ is small enough that quadratic products of its components may be neglected, the assumption $U^\nu{}_{\lambda\mu}=0$ gives a quantitative condition for weak fields. In the paper now under review the author observes that one might expect a generalization of his previous results to be obtained by seeking a structural condition that similarly becomes quantitative when cubic products of $k_{\lambda\mu}$ are are neglected. This procedure, however, leads to contradictions, and he makes instead the assumption that

$$(*)\qquad U^\nu{}_{\lambda\mu}=2p^\nu h_{\lambda\mu}-\delta_\lambda{}^\nu p_\mu-\delta_\mu{}^\nu p_\lambda,$$

where $p_\lambda=-(1/6)\ln g\neq0$. Geometrically, this means that the minimal autoparallels of $\Gamma^\nu{}_{\lambda\mu}$ (but only the minimal ones) are geodesics of $h_{\lambda\mu}$.

The first three chapters of the paper follow out the mathematical consequences of regarding (*) as a structural condition. Chapter I is mathematical. Chapter II contains a consideration of Maxwell's equations in vacuo, with the conclusions (i) that "the space dealt with in this chapter is an empty space"; (ii) that "the electromagnetic (gravitational) field may be expressed in terms of the gravitational (electromagnetic) field. The electromagnetic field is always present". In Chapter III the author finds that the world-line of a free particle in the unified field is not an autoparallel of $\Gamma^\nu{}_{\lambda\mu}$ under the assumptions until then made by him, but he shows that it is such an autoparallel if the assumptions are suitably modified. Among the autoparallels, only the world-lines of photons are also geodesics of $h_{\lambda\mu}$.

In the fourth and final chapter the structural condition (*) is converted into a quantitative one by the assumption that $k_{\lambda\mu}$ is "almost covariantly constant", thus replacing the assumption of the previous paper that quadratic products of $k_{\lambda\mu}$ could be neglected. Dealing quickly with the mathematical consequences of this new assumption, the author considers the application of his theory to celestial mechanics. Seeking the analogue of the Schwarzschild solution of classical relativity, he finds that the world-lines of photons are kinematically the same as in the classical theory, and therefore yield the same deflection of light and the same red-shift. But planetary orbits are not the same as in the classical theory, so that the numerical value for the advance of the perihelion of Mercury predicted by the unified theory may be expected to differ from that predicted by the classical theory. The solution of the equations of the orbits would require a knowledge of the electromagnetic field of the sun. *H. S. Ruse* (Leeds).

Lichnerowicz, André. Compatibilité des équations de la théorie unitaire du champ d'Einstein. J. Rational Mech. Anal. **3**, 487–521 (1954).

In the first chapter the author derives by means of variational principle the fundamental equations of the recent Einstein unified theory:

$$(1)\quad (a)\quad \partial_\omega g_{\lambda\mu}=\Gamma^\alpha{}_{\lambda\omega}g_{\alpha\mu}+\Gamma^\alpha{}_{\omega\mu}g_{\lambda\alpha},$$
$$(b)\quad \partial_\lambda \mathfrak{g}^{[\lambda\nu]}=0\quad(\mathfrak{g}^{\lambda\nu}=g^{\lambda\nu}\sqrt{\mathfrak{g}}),$$
$$(2)\qquad R_{\mu\lambda}=\partial_{[\mu}X_{\lambda]},$$

where $g^{\lambda\nu}$ is the inverse tensor to $g_{\lambda\mu}$, $\mathfrak{g}$ is the determinant of $g_{\lambda\mu}$, $R_{\lambda\mu}$ is the contracted curvature tensor of $\Gamma^\nu{}_{\lambda\mu}$ and X_λ is a vector. [Remark of the reviewer: For the use of variational principle in connection with (1) and (2) see also E. Schrödinger, Proc. Roy. Irish Acad. Sect. A. **51**, 163–171 (1947), 205–216 (1948); **52**, 1–9 (1948); these Rev. **9**, 310, 311; A. Einstein, Supplement to Appendix II of "The meaning of relativity, 4th ed.", Princeton, 1953; these Rev. **15**, 357.] Denoting by $\mathfrak{K}_\lambda{}^\nu$ a certain algebraic concomitant of $R_{\mu\lambda}$ and $g_{\lambda\mu}$, the author derives, again by means of variational principle, the four identities

$$(3)\qquad \partial_\lambda\mathfrak{K}_\mu{}^\lambda+\tfrac{1}{2}R_{\lambda\nu}\partial_\mu\mathfrak{g}^{\lambda\nu}=0\quad(\mathfrak{K}_\lambda{}^\nu=K_\lambda{}^\nu\sqrt{g}).$$

[Remark of the reviewer: Cf. also A. Einstein, Canadian J. Math. **2**, 120–128 (1950); these Rev. **11**, 548; S. N. Bose, C. R. Acad. Sci. Paris **236**, 1333–1335 (1953); these Rev. **14**, 915. In physical interpretations these identities are equivalent to the equations of motion of charged particles in the unified field. See V. Hlavatý, J. Rational Mech. Anal. **3**, 147–179 (1954), cited later on as C_2; these Rev. **15**, 654.] Denoting by $X_\lambda{}^\nu$ a certain algebraic concomitant of $\partial_{[\mu}X_{\lambda]}$ and $g_{\lambda\mu}$, the author uses later the tensor

$$(4)\qquad M_\lambda{}^\nu=K_\lambda{}^\nu-X_\lambda{}^\nu.$$

The second chapter represents one of the fundamental contributions to the unified field theory. It deals with the following (Cauchy) problem: Given the set of expressions

$$(5)\quad g_{ij},\ \mathfrak{g}^{[0i]},\ \mathfrak{g}^{(0\nu)}\partial_0 g_{ij},\ \partial_0\mathfrak{g}^{(0\nu)},\ X_\lambda\qquad(i, j, h=1, 2, 3;\ \nu, \lambda, \mu=0, \cdots, 4)$$

along the hypersurface S $(x^0=0)$ for which $g^{00}\neq0$, one has to find the solution $g_{\lambda\nu}$, X_λ of (1), (2) in the neighborhood of

S. The proof is based on the possibility of expressing the derivatives (with respect to x^0) of $g_{\lambda\mu}$, X_λ by means of (1), (2) along S. It boils down to the following steps. a) The expressions g_{ij}, $\mathfrak{g}^{[0i]}$, $\mathfrak{g}^{(0\lambda)}$ yield $g_{\lambda\mu}$. b) The functions $K_\lambda{}^0$ are expressible in terms of the first five sets of (5) (and their derivatives with respect to x^k) provided $g_{\lambda\mu}$ is a solution of (1b). c) The system (2) may be split in

$$(6)\qquad \text{(a)}\quad R_{(ij)}=0 \qquad\qquad \text{(b)}\quad R_{[\alpha\lambda]}=\partial_{[i}X_{\lambda]}$$

and

$$(7)\qquad \text{(a)}\quad M_\lambda{}^0=0$$

while (1b) may be written as

$$(6)\qquad \text{(c)}\quad \partial_\lambda\mathfrak{g}^{[\lambda i]}=0$$

and

$$(7)\qquad \text{(b)}\quad \partial_i\mathfrak{g}^{[i0]}=0.$$

Any solution $g_{\lambda\mu}$, X_λ of (6) satisfying (7) along S satisfies these latter conditions also in the neighborhood of S. (d) Assuming that the Cauchy data satisfy (7b) the equations (7a) yield $\partial_i X_j-\partial_j X_i$ provided $g_{\lambda\mu}$ is of the first class. [See Hlavatý, ibid. **1**, 539–562 (1952); these Rev. **14**, 416. The solution $g_{\lambda\mu}$ which is not of the first class is dealt with in Hlavatý, ibid. **3**, 103–146 (1954); these Rev. **15**, 654.] (e) The equations (6c) yield $\partial_0\mathfrak{g}^{[0i]}$, $\partial_{00}\mathfrak{g}^{[0i]}$. The equations (6a) and (6b) for $\lambda=j$ yield $\partial_{00}g_{ij}$. [The proof of this latter statement is based on the results of Hlavatý and Sáenz, ibid. **2**, 523–536 (1953); these Rev. **14**, 1132.] The condition (6b) for $\lambda=0$ yields $\partial_0 X_i$. The expressions (X_0 and) $\partial_0 X_0$ are not defined by the system (6), (7). The same is true for $\partial_{00}\mathfrak{g}^{(0\lambda)}$. To explain this latter fact the author proves that there are coordinate transformations which preserve on S the numerical values of the coordinates of points of S as well as the numerical values of the data of Cauchy and of $\partial_{00}g_{ij}$, $\partial_{00}\mathfrak{g}^{[0i]}$ while $\partial_{00}\mathfrak{g}^{(0\lambda)}$ acquires an arbitrarily preassigned numerical value. The solution of the problem is an unique one up to these coordinate transformations. (The author determines $\partial_0 X_0$ by a new condition. This corresponds to the condition imposed by the reviewer in C_2 and C_3 [see the preceding review] for defining the mass.) (f) The same method leads to higher derivatives (with respect to x^0) of $g_{\lambda\mu}$ and X_λ. (g) The condition $g^{00}\neq 0$ is essential for the proof. If ${}^*h_{\lambda\mu}$ denotes the inverse symmetric tensor to $g^{(\lambda\nu)}$ the requirement $g^{00}\neq 0$ is obviously equivalent to the requirement that S is not tangent to the cone ${}^*h_{\lambda\mu}dx^\lambda dx^\mu=0$. This suggests the identification of the gravitational field with ${}^*h_{\lambda\mu}$. [Remark of the reviewer: This new identification of the gravitational field leads to the same physical results in celestial mechanics as exhibited in C_2.] *V. Hlavatý.*

Hély, Jean. Sur une généralisation immédiate des équations d'Einstein. C. R. Acad. Sci. Paris **239**, 747–749 (1954).

The author proposes to replace two of the four sets of field equations of Einstein's generalized theory of relativity by introducing an ad hoc stress-energy tensor and a similar current vector. This step would seem to be in violation of the spirit of Einstein's generalization of the theory of relativity. *A. H. Taub* (Urbana, Ill.).

Corben, H. C. Aspetti fisici delle teorie unitarie. Rend. Sem. Mat. Fis. Milano **23** (1952), 152–163 (1953).

Attempts to develop a unified theory of gravitation and electromagnetism are reviewed and some details are given of those theories based on a five-dimensional continuum. It is pointed out that the formal success of these theories may be due to chance and that a new physical principle is needed before attempts in this direction can be useful. Such a principle may be related to the concept of charged observers, but until one knows exactly what is the nature of electric charge it is difficult to formulate such a theory. (From the author's summary.) *A. J. Coleman.*

Takeno, Hyôitirô. The problem of many bodies and the superposition of spherically symmetric space-times in general relativity. Progress Theoret. Physics **11**, 392–410 (1954).

It is suggested that a solution of the many-body problem in relativity might be obtained if the space-times associated with the separate bodies are superposed in some suitable way. The author does not here consider the general problem, but restricts himself to the case of static spherically symmetric space-times having a common centre of symmetry. These are superposed to give a static space-time which does not, in general, satisfy the Einstein field equations, and it is suggested that this is due to interaction between the component gravitational fields. *A. G. Walker.*

Bhattacharya, Shambhunath. The general theory of relativity and the expanding universe. Progress Theoret. Physics **11**, 613 (1954).

The author assumes that the space-time of the universe is a conformally flat space. The single function entering in the metric tensor is then determined from the field equation obtained by equating the scalar curvature tensor of the conformally flat space to the counteraction of the stress energy tensor of a perfect fluid with zero pressure.
A. H. Taub (Urbana, Ill.).

Infeld, L. Equations of motion and non-harmonic coordinate conditions. Bull. Acad. Polon. Sci. Cl. III. **2**, 163–166 (1954).

The author determines the equations of motion of two singularities of the solutions of the field equations of general relativity in the first (Newtonian) approximation. The authors' purpose is to show that the equations of motion can be derived by using a coordinate condition different from that used by Fock [Acad. Sci. U.S.S.R. J. Phys. **1**, 81–116 (1939); these Rev. **1**, 183]. The author states: "This is so because the coordinate condition has, in reality, nothing to do with the equations of motion, not only in the Newtonian but also in the next approximation after the Newtonian". *A. H. Taub* (Urbana, Ill.).

Ingraham, R. Conformal geometry and elementary particles. Proc. Nat. Acad. Sci. U. S. A. **40**, 237–240 (1954).

This note summarizes the main findings of a paper to appear shortly. This paper has to do with the kinematical properties of a theory in which the spheres of a flat four space with signature $+++-$ are taken to be points of a Riemannian 5-space of constant unit curvature. The particle-states, points in the 5-space, are labeled by the four coordinates of the center of the sphere and a fifth coordinate, of dimensions of the reciprocal of mass, labeling the radius of the sphere. The author proposes to discuss fields on these particle-states. The author states that the motion of test particles is described by geodesics in sphere space and gives the equations for these. *A. H. Taub* (Urbana, Ill.).

Callaway, Joseph. Mach's principle and unified field theory. Physical Rev. (2) **96**, 778–780 (1954).

The author gives a rough statement of Mach's principle of the relativity of inertia and recalls the existence of solu-

tions of Einstein's gravitational equations which appear to be inconsistent with it. He discusses difficulties about boundary conditions and about the interpretation of the energy-momentum tensor and concludes that without additional assumptions Mach's principle cannot be satisfied within general relativity theory.

The assumptions on which general relativity is based are not stated explicitly; it is not made clear, for example, whether the assumption (*): that the metric of space-time is everywhere of signature 2, is to be regarded as basic or additional, so that if the assumptions sufficient to imply Mach's principle were known, it is not clear whether they would from the author's point of view be generally agreed upon as "basic", like (*) or more arbitrary, and therefore regarded as "additional". It is implied that "additional" assumptions would be in some way unsatisfactory.

The author argues that if Mach's principle is abandoned then space-time and matter must enter into (non-quantum) physics with equal status, so that a unified field theory of some sort would be desirable. He suggests two possible approaches to such a theory.

This paper contains a number of statements which seem to the reviewer to require proof; in particular, it is asserted that the only cosmological solutions of Einstein's gravitational field equations which are consistent with Mach's principle are those in which the 3-surfaces cosmic time = constant, are of positive curvature. It appears to the reviewer that in general this restriction to positive curvature is unnecessary. *F. A. E. Pirani* (Dublin).

Zeuli, Tino. Sul movimento di fluidi veloci. Atti Sem. Mat. Fis. Univ. Modena 6 (1951–52), 3–15 (1953).

Using the notation of homographies the author obtains the usual relativistic equations of motion of a perfect fluid and (in the special theory) discusses the rate of change of vorticity, Lagrangian coordinates, and the integral of Bernoulli. *J. L. Synge* (Dublin).

Layzer, David. On the significance of Newtonian cosmology. Astr. J. 59, 268–270 (1954).

The author criticises the Newtonian cosmological theory of E. A. Milne and W. H. McCrea [Quart. J. Math., Oxford Ser. 5, 64–72, 73–80 (1934)], claiming that it is incompatible with the Newtonian concept of gravitation which it tries to incorporate, and that all Newtonian derivations of the Einstein-Friedmann equations are invalid. But by virtue of two relativistic theorems, due essentially to H. Bondi, the mathematical apparatus of Newtonian theory may be used to obtain exact solutions of cosmological problems belonging to a certain well-defined class. *J. L. Synge.*

Synge, J. L. Note on the Whitehead-Rayner expanding universe. Proc. Roy. Soc. London. Ser. A. 226, 336–338 (1954).

On the basis of Whitehead's gravitational theory, C. B. Rayner [same Proc. 222, 509–526 (1954); these Rev. 15, 835] obtained the fundamental (gravitational) form for a uniformly expanding world-model as a particular case of a non-static spherically symmetric system. In the present paper a direct calculation in flat Minkowski space-time with coordinates x_r ($r=1$ to 4, $x_4=ict$) leads much more simply to the fundamental form

$$g_{mn}dx_m dx_n=\left(1+\frac{A}{R}\right)dx_n dx_n+\frac{4A}{R}dR^2,$$

where A is a particular constant of the model and $R^2=-x_nx_n$. Rayner's form is obtained by transforming to polar coordinates. *G. J. Whitrow* (London).

Whitrow, G. J. E. A. Milne's scales of time. British J. Philos. Sci. 5, 151 (1954).

Smith, E. A. New method of calculating gravitational fields in the universe. Revista Ci., Lima 55, 237–254 (1953).

MECHANICS

✶Artobolevskiĭ, I. I. Teoriya mehanizmov i mašin. [Theory of mechanisms and machines.] 3d ed. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 712 pp. 14.60 rubles.

Except for minor corrections this edition is the same as the second edition [1951; these Rev. 15, 566].

Baranov, G. G. Classification, structure, kinematics and kinetostatics of plane mechanisms with pairs of the first kind. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov 12, no. 46, 15–39 (1952). (Russian)

L. V. Assur [Issledovanie ploskih steržnevyh mehanizmov s točki zreniya ih struktury i klassifikacii, Izdat. Akad. Nauk SSSR, Moscow, 1952, reprinted from Izv. Sankt-Peterburg. Politekh. Inst. 20, 329–385 (1912), 581–635 (1913); 21, 187–283 (1914)] designated linkages to be of the kth class if they consisted of $2k$ members joined by $3k$ pinned rotary joints. The addition of another member, with no increase in the number of joints, converts the linkage into a rigid truss of the kth class. Let the truss have n_2 bars, n_3 triangles, n_4 quadrilaterals, etc., where the subscript is the number of pins in a member. Then

$$n_2+n_3+n_4+\cdots=2k+1 \quad\text{and}\quad 2n_2+3n_3+4n_4+\cdots=6k,$$

from which $n_2=3+n_4+2n_5+3n_6+\cdots$. Therefore, in a truss without redundant constraints, there are at least 3 bars. The maximum possible number of triangles is given by $n_3=2k-2$, obtained by letting $n_2=3$ and $n_4=n_5=n_6=\cdots=0$.

This paper develops and exhibits all the 26 trusses of the fourth class. By removal of single members of these trusses in different ways, and elimination of duplication due to symmetry, the totality of the 161 Assur linkages of the fourth class is derived. Previous work had carried out this development only up to the third class.

Graphical methods are given for the determination of the positions, velocities, accelerations and forces in the three types of connections of a member with its neighbors ~~discussed~~. These types are two rotary pivots, a pivot and a slide, or two slides. *M. Goldberg.*

✶Groeneveld, Bernard. Geometrical considerations on space kinematics in connection with Bennett's mechanism. Thesis, Technische Hogeschool te Delft, 1954. 112 pp.

Consider a skew hinged four-bar mechanism in three-space. The angle between the hinges in a bar is called the twist. This mechanism is movable only if the opposite sides are equal. Then it follows as a consequence that the sines of the twists are proportional to the lengths of the bars. This remarkable mechanism was disclosed by G. T. Bennett [Engineering 76, 777–778 (1903)] who called it a skew iso-

gram. His later paper [Proc. London Math. Soc. (2) **13**, 151–173 (1914)] presented many interesting properties of the skew isogram, some without proofs. The principal aim of the present dissertation is to supply these missing proofs.

In addition to Bennett's theorems, the work of subsequent writers on Bennett's mechanism is displayed. These include the derivation of the algebraic relations among the angles of the isogram, the fourth-degree ruled surface generated by the bar opposite a fixed bar and various associated quadrics.

A good bibliography of eighteen references is marred by the omission of important Russian references [Dimentberg and Shor, Akad. Nauk SSSR. Ž. Prikl. Mat. Meh. (N.S.) **4**, no. 3, 111–118 (1940); Dimentberg, The determination of the positions of spatial mechanisms, Izdat. Akad. Nauk SSSR, Moscow, 1950; these Rev. **12**, 867] and the applicable papers of the reviewer [Trans. Amer. Soc. Mech. Eng. **65**, 649–661 (1943); J. Math. Physics **25**, 96–110 (1946); these Rev. **6**, 74; **8**, 99]. In the bibliography, M. F. E. Myard should be F. E. Myard; J. Lond. Math. Soc. should be Proc. Lond. Math. Soc. *M. Goldberg.*

Bottema, O. Zur Kinematik des Rollgleitens. Arch. Math. **6**, 25–28 (1954).

Grüss [Z. Angew. Math. Mech. **31**, 97–103 (1951); these Rev. **13**, 80] introduced the study of the kinematics of the motion of a plane curve k over another fixed plane curve K so that the ratio of the lengths of the traversed arcs of the two curves is a constant λ. The pole curves were derived. Müller [Arch. Math. **4**, 239–246 (1953); these Rev. **15**, 171] simplified and extended the work. In particular, he solved the inverse problem, namely, that of determining the curves k and K when λ and the pole curves are given. The present paper examines Muller's work more closely, showing that for $\lambda=1$, the unique pole curves are obtained directly. However, for each $\lambda\neq 1$, the pole curves are an infinite set satisfying differential equations of the first order; the tangents satisfy equations of the second order, while the curvatures satisfy equations of the third order. The foregoing differential equations are derived and solved explicitly.

M. Goldberg (Washington, D. C.).

Tolle, O. Über Anwendungen von "Plänen relativer Normalbeschleunigungen" in der Getriebedynamik. (Trägheitspol einer ebenbewegten Scheibe und Krümmungsmittelpunkt von Koppelkurven.) Ing.-Arch. **22**, 227–236 (1954).

The force on a moving plane body may be determined by the use of the angular velocity and acceleration of the body, its mass and moment of inertia, and the locations of the center of gravity and center of oscillation. A graphical construction which made use of the "plane of relative normal acceleration" was discussed by M. Tolle [Z. Verein. Deutsch. Ingenieure **76**, 799–800 (1932)] and others. The present paper summarizes the construction and shows several applications. In particular, the radius of curvature of the paths of selected points of the body may be determined. *M. Goldberg* (Washington, D. C.).

Stoppelli, Francesco. Una rappresentazione geometrica dei rotori e sue applicazioni all'astatica. Ricerche Mat. **3**, 95–107 (1954).

A rotation of a rigid body about a fixed point is given by an orthogonal matrix $\|\gamma_{ij}\|$. In view of the fact that there are ∞^3 rotations the author considers γ_{11}, γ_{22} and γ_{33} as given and represents the rotation by the image point $P\equiv(\gamma_{11}, \gamma_{22}, \gamma_{33})$ in three-dimensional euclidean space. The points P are shown to be the points within or on the border of a certain tetrahedron T. Each inner point of T corresponds to eight rotations. An application is given to a problem of astatics. A system of forces is in equilibrium and each force subjected to a rotation R. The author proves that the new system is again in equilibrium if the image of R belongs to a certain parabolic cylinder circumscribing T.

O. Bottema (Delft).

Herivel, J. W. A general variational principle for dissipative systems. Proc. Roy. Irish Acad. Sect. A. **56**, 37–44 (1954).

In a dynamical system, described in the classical notations, set

$$a_s=\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s}-\frac{\partial T}{\partial q_s}.$$

If the generalized forces are of the form $Q_s(q)-\partial F/\partial\dot{q}_s$, $F=F(q,\dot{q})$, then the equations of motion are

$$(*)\qquad a_s=Q_s-\partial F/\partial\dot{q}_s.$$

Set

$$G\equiv\sum_s\dot{q}_s(a_s-Q_s)+F.$$

Hold the a_s and the q_s constant, but allow the $\dot{q}_s$ to vary. Then the equations of motion (*) are equivalent to $\partial G/\partial\dot{q}_s=0$. This obvious fact the author calls a variational principle for dissipative systems. He states that J. L. Synge showed him a proof of it, and he gives in detail a proof for a special case. He notices that in general F is not the rate of working against the frictional forces, which is $\sum\dot{q}_s\partial F/\partial\dot{q}_s$. [Obviously the two are equal if and only if F is homogeneous of degree 1 in the $\dot{q}_s$, and they stand in the ratio n if F is homogeneous of degree n.] This the author observes in a special case and concludes that therefore frictional heat is "of less importance . . . than might at first be supposed."

Guided by analogy to the foregoing, the author writes down

$$\delta\int\rho[\mathbf{v}\cdot\mathbf{a}+\dot{U}+\dot{V}]d\tau=0,$$

where $\mathbf{v}$ and $\mathbf{a}$ are velocity and acceleration, U and V are the internal and extrinsic energies, and the dot denotes the material derivative. For fields $\mathbf{v}$ having the same acceleration, he verifies that the corresponding Euler equation, when the entropy is substantially constant, is the dynamical equation for inviscid fluids. For viscous fluids he sets up a similar formal variation with $\frac{1}{2}t_{ij}v_{j,i}$ added to the integrand, where t_{ij} is the tensor of viscous stresses, assumed linear. By considering the surface terms, he obtains an expression for the resistance experienced by a sphere. The paper closes with a statement of the result from the beginning, expressed in terms of electric currents in a network.

[For recent work on variational principles in fluids, see L. Lichtenstein, Grundlagen der Hydromechanik, Springer, Berlin, 1929, Chap. 9, §§1–2; van den Dungen, Colloques Internat. Centre Nat. Recherche Sci., Paris, no. **14**, 88–95 (1949); these Rev. **11**, 622; Taub, Proc. Symposia Appl. Math., v. 1, Amer. Math. Soc. New York, 1949, pp. 148–157; these Rev. **11**, 221; Miche, J. Math. Pures Appl. (9) **29**, 151–179 (1949); these Rev. **11**, 62; Gerber, Ann. Inst. Fourier Grenoble **1**, 157–162 (1950); J. Math. Pures Appl. (9) **32**, 79–84 (1953); these Rev. **11**, 696; **15**, 173; Delval, Acad. Roy. Belgique Bull. Cl. Sci. (5) **36**, 639–648 (1950); these Rev. **12**, 762; Moreau, J. Math. Pures Appl. (9) **31**,

355–375 (1952); **32**, 1–78 (1953), see §7C, these Rev. **14**, 1027; the reviewer, footnote 1 to §94 of The kinematics of vorticity, Indiana Univ. Press, Bloomington, 1954.]

C. Truesdell (Bloomington, Ind.).

Dubreil-Jacotin, M. L. Quelques remarques sur le problème de la balançoire. Rev. Gén. Sci. Pures Appl. **61**, 264–271 (1954).

The author discusses the mechanics of a person setting a swing, in which he is standing, into large oscillations. The basic idea is that the system is equivalent to a pendulum which is more or less abruptly shortened as it passes through the vertical, and is more or less abruptly lengthened as it passes through the extremes of its displacement. The question of whether or not the oscillations can build up into a rotation in a fixed sense, and the question of the optimum performance of the person in order to achieve large oscillations quickly, are considered. *L. A. MacColl.*

Winter, R. Comment simplifier les calculs balistiques sur roquettes lancées du sol par l'intermédiaire du "projectile passif équivalent". Mém. Artillerie Française **28**, 545–611 (1954).

The practice of ballistic computation has been characterized at nearly every historical stage of its development by adoption of simplifying conventions which replace admitted variables by serviceable constant mean values. Advances in theoretical accuracy of computation have been largely due to taking into explicit account known changing features in the dynamic system, rather than attempting to economize in labor by treating incident variables as ballistically equivalent to their mean constant values. Of course these advances have cost much in added computational labor. In this country many refinements have become feasible only by virtue of the availability of reliable high-speed computing devices. This article develops the idea adequately expressed by its title. It seeks to replace the two- or three-step theory of an actual ground-to-ground rocket, by that of an "equivalent passive projectile" which after the end of action of the propellant will perform in a manner "geometrically and kinematically" identical to that of the given rocket. The details are worked out in great elaboration with numerical tables, using, however, series developments carried only through terms of second order. No indications are offered as to the experimental agreement between this theory and practice. The well-known French GHM theory for particle trajectories is taken as basic. *A. A. Bennett.*

Hydrodynamics, Aerodynamics, Acoustics

***Landau, L. D., i Lifšic, E. M. Mehanika splošnyh sred. [The mechanics of continuous media.] 2d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 788 pp. 15.35 rubles.

The book treats the theory of motion of fluids and gases (hydrodynamics) and solid material (elasticity). The chapters on hydrodynamics are: 1) Ideal fluids; 2) Viscous fluids; 3) Turbulence; 4) Boundary layer; 5) Heat conduction; 6) Diffusion; 7) Surface phenomena; 8) Sound; 9) Shock waves; 10) One-dimensional gas flow; 11) Surfaces of discontinuity; 12) Plane gas flow; 13) Flow about a finite body; 14) Combustion; 15) Relativistic hydrodynamics; 16) Superfluidity, in all 628 pages. Elasticity is confined to infinitesimal strain and comprises: 1) Basic equations; 2) Equilibrium of rods and plates; 3) Elastic waves; 4) Heat conduction, in all 156 pages.

To cover such a vast field the treatment has to be concise. Nevertheless the exposition is a model of clarity. Approximate and empirical methods are not treated and to the authors' credit they never seem to lose sight of the physical background. The book is clearly intended for physicists, but, in spite of the number of worked problems, it is difficult to see to what class of worker the book will appeal. In any one division there is too little for the specialist, while the whole could prove indigestible to the tyro. As giving an overall picture, it is a magnificent effort.

L. M. Milne-Thomson (Greenwich).

Marchetti, Luigi. Sul moto di un corpo rigido in un gas indefinito. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **15**, 274–278 (1953).

The author points out that in the class of steady flows having the same density field Kelvin's theorems of minimum energy and uniqueness of potential flow hold. [The reviewer regards this fact as well known and has included a proof of a variant of the latter theorem in §37 of The kinematics of vorticity, Indiana Univ. Press, Bloomington, 1954.] The author suggests that one could solve the problem of potential flow by this means: using measured or otherwise conjectured values of the density, set up the velocity potential as a linear combination of known solutions of the potential equation with this given density, then determine the coefficients so as to minimize the kinetic energy.

C. Truesdell (Bloomington, Ind.).

Bazer, J., and Karp, S. N. Potential flow through a conical pipe with an application to diffraction theory. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University Research Rep. No. **EM-66**, ii+70 pp. (1954).

"The primary object of this report is to obtain an exact solution of the following potential problem: find an axially symmetric solution of the potential equation having a vanishing normal derivative on the surface of a semi-infinite conical pipe with a circular aperture and having a prescribed behavior at infinity. The solution may immediately be interpreted as the velocity potential of a steady-state irrotational flow of a non-viscous incompressible fluid through a rigid conical pipe with a circular aperture. Using this interpretation, the solution is employed to derive, for suitable excitation, an approximate expression for the far field of the corresponding boundary-value problem involving the diffraction of sound." (From the author's abstract.)

It is difficult to give an adequate account of the report in a brief space. The potential problem is solved by using a variant of the Wiener-Hopf procedure. To obtain the approximate far field in the diffraction problem, Rayleigh's static method is used, based on the fact that when the wave length is large compared with the dimensions of the aperture, the solution near the aperture is represented approximately by the solution of the corresponding potential problem.

E. T. Copson (St. Andrews).

Stewartson, K. On the free motion of an ellipsoid in a rotating fluid. Quart. J. Mech. Appl. Math. **7**, 231–246 (1954).

The author investigates the stability under small disturbances of the unconstrained motion of a spheroid in a rotating fluid. The spheroid is in relative equilibrium when

rotating about its axis of symmetry (which is parallel to the axis of rotation of the fluid) with the same angular velocity as the fluid. It is then displaced and allowed to move under the influence of surface pressures alone. As in an earlier work by the author [same J. 6, 141–162 (1953); these Rev. 14, 1138] the hydrodynamic forces and moments are calculated on the assumption of small velocities relative to the undisturbed rotating fluid. When inserted in the equations of motion for the body these forces and moments provide the means for analyzing the stability of the motion. The calculations show that if the density of the spheroid relative to the fluid is σ, "it will be in stable equilibrium for translational disturbances if $\sigma \leqq 1$, and for rotational disturbances if $\sigma \geqq 1$ in the case when it is oblate, or if $\sigma \leqq 1$ in the case when it is prolate. Further, if $1>\sigma>1/9$, it will have a single period of translational oscillation, while if $\sigma<1/9$ it will have two such free rotational periods; and if stable, it may also have a free period of rotational oscillation." (Quotation from the author's summary.) The paper describes several experiments whose results were in substantial qualitative agreement with the theory.

D. Gilbarg (Stanford, Calif.).

v. Krzywoblocki, M. Z. On the stability of Bénard-Kármán vortex street in compressible fluids. II. Acta Physica Austriaca 8, 370–387 (1954).

The author continues his discussion of the stability of the Kármán vortex street in compressible fluids [cf. same Acta 7, 283–298 (1953); these Rev. 15, 479]. Since he again makes the tacit assumption that the individual vortex potentials are linearly superposable, it is difficult to see the precise meaning of his work for compressible fluids. *D. Gilbarg.*

***Garabedian, P. R. An example of axially symmetric flow with a free surface.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 149–159. Academic Press Inc., New York, 1954. $9.00.

This paper presents an example in closed form of a steady-state axially symmetric flow possessing a constant-pressure free surface. The construction is based on an inverse procedure: The author first gives a formula for an axially symmetric flow having an arbitrary analytic curve C in a meridian plane as free boundary. The desired example is then obtained by choosing C to be the free boundary of a suitable plane free-boundary flow; the resulting flow is shown to be defined in the large and to exhibit reasonable behavior on the axis of symmetry and at infinity. The particular plane flow used is of some interest in itself, since it is past an isolated object bounded entirely by free surfaces.

J. B. Serrin (Minneapolis, Minn.).

***Busemann, A. Minimum virtual mass.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 25–40. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

Riabouchinsky [C. R. Acad. Sci. Paris 185, 840–841 (1927)] and others have observed that bodies achieving minimum virtual mass in a uniform flow are bounded in part by constant speed streamlines. The author of the present paper describes several such minimum virtual mass free surface flows, these corresponding to different polygonal fixed boundaries presumably suggested by engineering applications. The discussion is entirely qualitative.

D. Gilbarg (Stanford, Calif.).

Benjamin, T. B., and Ursell, F. The stability of the plane free surface of a liquid in vertical periodic motion. Proc. Roy. Soc. London. Ser. A. 225, 505–515 (1954).

When a vessel containing liquid is vibrated vertically, standing waves on the free surface are often observed. Faraday and Rayleigh observed such waves, having half the frequency of the vessel vibration. Matthiessen, on the other hand, found synchronous standing waves. The author shows that the amplitude $a_m(t)$ of the standing waves associated with a "normal mode" satisfy the Mathieu equation

$$d^2a_m/dT^2+(p_m-2q_m\cos 2T)a_m=0, \quad T=\omega t/2,$$

if the vertical vibration of the vessel is $f\cos\omega t$. Here p_m and q_m are associated with free oscillations and forced oscillations, respectively. The half-frequencies observed by Faraday and Rayleigh are shown to be associated with the instabilities of the Mathieu equation known to occur in the neighborhood of $p=1$, and the synchronous standing waves observed by Matthiessen with the instabilities in the neighborhood of $p=4$. *G. Birkhoff* (Cambridge, Mass.).

Birkhoff, Garrett. Note on Taylor instability. Quart. Appl. Math. 12, 306–309 (1954).

G. I. Taylor has discussed the stability under normal acceleration of a plane interface separating two fluids of different density [Proc. Roy. Soc. London. Ser. A. 201, 192–196 (1950); these Rev. 12, 58]. His main conclusion is: the interface is unstable when the light fluid is accelerated towards the dense fluid and (presumably) stable when the reverse holds. This conclusion has applications to gas-filled underwater explosion bubbles. Its applicability to small vapor-filled (i.e. constant pressure) cavities is, however, less clear. The purpose of this note is to show that, in spite of the fact that the denser liquid is being accelerated towards the lighter vapor, collapsing bubbles are unstable, and that this result is unaffected by surface tension (though it may be affected by viscosity or thermodynamic considerations). The proof of this fact depends on a study of the stability near $t=0$ of differential equations of the form

$$t^nx^{(n)}+a_1t^{n-1}x^{(n-1)}+\cdots+a_nx=0, \quad x^{(k)}=d^kx/dt^k.$$

Since $t=0$ is a regular singular point of this equation, the type of instability arising is algebraic and not of the exponential kind usually considered. *J. B. Serrin.*

Friedrichs, K. O., and Hyers, D. H. The existence of solitary waves. Comm. Pure Appl. Math. 7, 517–550 (1954).

Les auteurs nous apportent la réponse à la question de l'existence analytique de l'onde solitaire. Dans leur travail ils donnent une démonstration directe de cette existence que nous allons résumer. Dans un travail récent [Akad. Nauk Ukrain. RSR. Zbirnik Prac' Inst. Mat. 1946, no. 8, 13–67 (1947); Amer. Math. Soc. Translation no. 102 (1954); ces Rev. 14, 102; 15, 906] Lavrent'ev a étudié l'existence de l'onde solitaire en considérant celle-ci comme limite d'ondes périodiques dont la période croît indéfiniment.

Soient, H_0 la hauteur d'eau à l'infini et V_0 la vitesse de propagation de l'onde. On suppose $gH_0/V_0^2<1$ et on pose $gH_0/V_0^2=\exp(-3a^2)$. Le problème de l'onde solitaire peut alors s'énoncer ainsi: Dans la bande $\mathfrak{B}$, $0<\psi<1$, $-\infty<\varphi<+\infty$, du plan $\zeta=\varphi+i\psi$, déterminer une fonction analytique $\omega=\theta+i\tau$, non identiquement nulle, qui satisfasse aux conditions aux limites suivantes:

$$\begin{aligned} &(1) \qquad \theta_\psi-\theta=e^{-3\tau}\sin\theta-\theta=F(\theta,\tau) \quad \text{sur} \quad \psi=1\\ &(2) \qquad \theta=0 \quad \text{sur} \quad \psi=0\\ &(3) \qquad \tau\to a^2 \quad \text{si} \quad \varphi\to\pm\infty. \end{aligned}$$

Les auteurs ramènent ce problème harmonique aux limites à la résolution d'une équation fonctionnelle non linéaire. Pour cela ils ont réussi à former une fonction de Green $J(\zeta, \zeta')$, qui permet de construire une fonction $\omega=\theta+i\tau$, analytique dans $\mathfrak{B}$, et vérifiant ces conditions à la frontière:

$$(4)\qquad \theta_\psi-\theta=f(\varphi)\quad \text{sur}\quad \psi=1,$$

$$(5)\qquad \theta=0\quad \text{sur}\quad \psi=0,$$

où $f(\varphi)$ est une fonction fixée, ayant certaines propriétés de régularité à l'infini. Sur $\psi=1$, ω est alors donnée par $\omega=\Omega[f(\varphi)]$, où Ω est un opérateur intégral linéaire. De plus, si l'on suppose que $f(\varphi)$ est impaire et que

$$(6)\qquad \int_{-\infty}^{+\infty} \varphi f(\varphi)d\varphi=0,$$

il résulte de la forme de $J(\zeta, \zeta')$ que $\Omega[f(\varphi)]\to 0$ si $\varphi\to\pm\infty$.

Le problème de l'onde revient donc à la recherche des solutions non nulles de l'équation

$$(7)\qquad \theta(\varphi, 1)+i\tau(\varphi, 1)=\Omega[F(\theta(\varphi, 1), \tau(\varphi, 1))]+ia^2$$

pour lesquelles θ est impaire, tandis que τ est paire, et qui satisfait en outre à la condition

$$(8)\qquad \int_{-\infty}^{+\infty} \varphi F(\theta(\varphi, 1), \tau(\varphi, 1))d\varphi=0.$$

La démonstration de l'existence d'une solution de ce dernier problème repose sur un changement de variable et d'inconnues. Les auteurs posent $\hat{\varphi}=a\varphi$, $a^2\hat{\theta}=\theta$, $a^2\hat{\tau}=\tau$, et (7) est ainsi transformée en une équation en $\hat{\theta}(\hat{\varphi})$ et $\hat{\tau}(\hat{\varphi})$, qui pour $a=0$ possède une solution, non nulle, vérifiant la condition déduite de (8). C'est cette solution qui va être prolongée pour a suffisamment petit.

En introduisant un espace fonctionnel convenable B, l'équation (7) se met sous la forme

$$(9)\qquad G(\omega, a)=0$$

où ω est un point de B, et où la transformation G est continue sur B et admet une différentielle de Fréchet $\delta G(\omega_1, a, \delta\omega)$, également continue, au point ω_1 qui est la solution non nulle de l'équation

$$(10)\qquad G(\omega, 0)=0.$$

Les auteurs montrent que la solution générale de l'équation linéaire, non homogène, en $\delta\omega$

$$(11)\qquad \delta G(\omega_1, 0, \delta\omega)=\delta\zeta,$$

$\delta\zeta$ étant un élément arbitraire de B, est de la forme

$$\delta\omega=K[\delta\zeta]+c\nu_1$$

où K est une fonctionnelle opérant sur B, c une constante numérique arbitraire, et ν_1 une fonction définie au moyen de ω_1. Cela leur permet de conclure que pour a assez petit (9) admet une solution $\omega(a, c)$ dépendant du paramètre c.

Il suffit alors que l'on puisse déterminer c de telle manière que soit satisfaite la condition

$$(12)\qquad \Gamma(a, c)=\int_{-\infty}^{+\infty} \hat{\varphi}F(\hat{\theta}(\hat{\varphi}, a, c), \hat{\tau}(\hat{\varphi}, a, c))d\hat{\varphi}=0.$$

Les propriétés de $\Gamma(a, c)$ au voisinage des valeurs $a=c=0$ et les résultats de la théorie des fonctions implicites montrent qu'il en est bien ainsi pour a suffisamment petit.

R. Gerber (Toulon).

Tan, H. S. On motion of submerged cylinder. J. Aeronaut. Sci. **21**, 848–849 (1954).

Some remarks on the history of the problem of determining the fluid motion associated with the steady motion of a submerged circular cylinder at a constant depth below the free surface of a heavy fluid. The author states that a solution given by Sretenskiĭ [Trudy Central. Aero-Gidrodinam. Inst. no. **346** (1938), translated in NACA Tech. Memo. no. **1335** (1952); these Rev. **14**, 508] is in error.

J. V. Wehausen (Providence, R. I.).

Cheng, H. K., and Rott, N. Generalizations of the inversion formula of thin airfoil theory. J. Rational Mech. Anal. **3**, 357–382 (1954).

The paper is concerned with the determination of two-dimensional incompressible fields of flow for certain types of boundary conditions on the real axis. More particularly, a number of points are given on the real axis, and the velocity components u and v are specified alternately on the segments determined by these points. Types of singularities which are likely to arise in the applications (algebraic branchpoints, poles) are admitted at the endpoints of the segments. The determination of the homogeneous solutions (for which $u=0$ or $v=0$ on alternate segments) forms an essential part of the analysis, and the treatment of the general case is based on these homogeneous solutions. An inversion formula is obtained which reduces to a formula due to Munk for the special case of a single segment (aerofoil). A special technique is required in order to adjust the conditions at infinity.

A. Robinson (Toronto, Ont.).

Woods, L. C. The lift and moment acting on a thick aerofoil in unsteady motion. Philos. Trans. Roy. Soc. London. Ser. A. **247**, 131–162 (1954).

The intention here is to provide a theory of thick airfoils in unsteady motion, wherein the shed vortices are carried downstream with local, rather than stream, speed and the Kutta-Joukowski condition at the trailing edge can be relaxed to provide for boundary-layer effects. The function $f=\ln(dz/dw)$ is taken as the dependent variable, where w is the complex potential and z the complex coordinate of the physical plane. The flow is taken to be a small perturbation of the steady flow past the profile at zero lift. The fundamental relations used are (i) the integral expression for f in terms of the velocity around the airfoil and the wake-vortex strength, (ii) statement of vanishing disturbance at infinity, (iii) condition that total circulation about airfoil and wake remains equal to zero, and (iv) statement that wake vortices move with local velocity of the steady flow. These leave the circulation undetermined, since the trailing-edge condition has not been applied; the coordinate of the rear stagnation point remains a parameter in the formulas derived for lift, drag, and moment. The proposal made is to introduce this coordinate empirically, e.g. from steady-flow data concerning the airfoil, permitting infinite velocities to appear at the sharp trailing edge. This is a familiar procedure in steady-airfoil theory. [The author's statement that the Kelvin circulation theorem is used "instead of" the usual trailing-edge condition is misleading, since that theorem is always used in unsteady-airfoil theory in addition to the Kutta-Joukowski condition.]

When the general formulas are applied to the case of harmonic oscillations, it is specifically assumed that this coordinate oscillates harmonically in phase with and proportional in magnitude to the relative, instantaneous incidence at the trailing edge; this provides, rather arbitrarily,

a method for carrying over the steady-flow airfoil properties. The numerical results, i.e. lift and moment due to plunging and pitching, differ appreciably from the usual results of thin-airfoil theory by the presence of both thickness effects and terms proportional to the amplitude of the rear-stagnation-point oscillation. A comparison with experimental results of Bergh and van de Vooren [Nationaal Luchtvaart Laboratorium, Amsterdam. Reports **F. 103**; **F. 104** (1952)] is made. In most respects it appears that the trends of the new theoretical results, compared to thin-airfoil results, are correct. *W. R. Sears* (Ithaca, N. Y.).

Mendelson, Alexander, and Carroll, Robert W. Lift and moment equations for oscillating airfoils in an infinite unstaggered cascade. NACA Tech. Note no. **3263**, 46 pp. (1954).

The integral equation for the pressure jump across a representative airfoil in the cascade is formulated. By assuming phase differences of 90°, 180°, or 360° between adjacent airfoils this integral equation is transformed to the well known form having the kernel $(x-\xi)^{-1}$ and inverted. The results are applied to in-phase bending oscillations, and it is shown that cascading greatly reduces the aerodynamic damping. The in-phase problem has been solved previously by Billington [Div. Aeronaut., Counc. Sci. and Indust. Res. Australia, Rep. **E63** (1949)], but the author implies that his results are more convenient for numerical calculations. The author also notes that the case of arbitrary phasing has been treated approximately by Sisto [Thesis, Mass. Inst. Tech., 1952]. *J. W. Miles* (Los Angeles, Calif.).

van de Vooren, A. I. An approach to lifting surface theory. National Luchtvaartlaboratorium, Amsterdam. Report **F. 129**, 14 pp. (1953).

The paper discusses the lift of a wing in steady subsonic flow. By means of a coordinate transformation the author is able to extend the lifting surface theory, given by Reissner [Proc. Nat. Acad. Sci. U. S. A. **35**, 208–215 (1949); these Rev. **10**, 753] to wings that are both tapered and swept. A series expansion is introduced that gives the chord-wise vorticity distribution; the coefficients in the series are functions of only one variable. The infinite series is approximated by the first r terms, the truncation error being minimized by a suitable choice of the positions of chordwise pivotal points. When the aspect ratio of the wing is small the number of chordwise pivotal points can be large and the number of spanwise stations small; the author's method is ideal for such wings. However, in the opinion of the reviewer, the method due to Multhopp [R. A. E. Report Aero. **2353** (1950)] probably is preferable to the author's when the aspect ratio is not small. It would be interesting to see the present theory extended to unsteady motion and to the case when central surfaces are present; such an investigation has already been made using Multhopp's method [H. C. Garner, Aeronaut. Res. Council, rep. **15,096** (July, 1952)]. *G. N. Lance* (Los Angeles, Calif.).

Timman, R., van de Vooren, A. I., and Greidanus, J. H. Aerodynamic coefficients of an oscillating airfoil in two-dimensional subsonic flow. J. Aeronaut. Sci. **21**, 499–500 (1954).

The authors' previous work [same J. **18**, 797–802, 834 (1951); these Rev. **13**, 880] on the problem of the title contained numerical errors, as observed by Fettis [ibid. **19**, 353–354 (1952)]. Corrected values of the complex force and moment coefficients associated with oscillations of simple translation or rotation are presented in the present paper. The authors state that corrected values of those derivatives associated with control-surface motion may be obtained from them. *J. W. Miles* (Los Angeles, Calif.).

Castoldi, Luigi. Teoremi di Bernoulli per fluidi comprimibili viscosi. Atti Accad. Ligure **9** (1952), 215–221 (1953).

The author seeks to generalize the theorem of Bernoullian type obtained by the reviewer [Physical Rev. (2) **77**, 535–536 (1950); these Rev. **11**, 472]. His line of argument has meanwhile been put in yet more general form by the reviewer [Chapter VII, especially §74, of "The kinematics of vorticity", Indiana Univ. Press, Bloomington, 1954]. The author considers special cases when the result is particularly simple. He points out that through each point there pass in general infinitely many curves on which the classical Bernoullian expression is constant. If the viscosity approaches zero, any one of these curves will approach a curve on Lamb's surfaces in classical hydrodynamics.

The author takes issue with the reviewer on two points. One of these is a matter of taste or as yet undevised experiment: the author regards the mean pressure, in any homogeneous viscous fluid, as an assignable function of density only, while the reviewer is not aware of any reason or evidence in favor of this opinion. The second relates to the special choice of Bernoullian lines made by the reviewer. This special choice determines the Bernoullian lines from the instantaneous velocity field alone, and the reviewer's objective was to show that such lines exist and to find them. This advantage is not preserved in the author's treatment, which in effect defines Bernoullian lines as any curves on the surfaces where the classical Bernoullian expression is constant. *C. Truesdell* (Bloomington, Ind.).

Borgnis, F. E. Über die Bewegungsgleichung und den Impulssatz in viskosen und kompressiblen Medien. Acustica **4**, 407–410 (1954).

Textbook material regarding the Stokes relation. [For bibliography and discussion of the points raised by the author, see Truesdell, J. Rational Mech. Anal. **1**, 125–300 (1952), §61A; **2**, 593–616, 643–741 (1953), §4; Z. Physik **131**, 273–289 (1952); Proc. Roy. Soc. London. Ser. A. **226**, 59–65 (1954); these Rev. **13**, 794; **15**, 178, 757; **14**, 231; **16**, 298.] *C. Truesdell* (Bloomington, Ind.).

***Kuerti, G. On a class of spherically symmetric flows.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 160–169. Academic Press Inc., New York, 1954. $9.00.

This paper gives a fairly complete description of motions for which the rate-of-strain tensor is spherical, particular attention being paid to those which satisfy the Navier-Stokes equations with barotropic equations of state. The Stokes relation, connecting the two viscosity coefficients, is assumed. It is of some interest that more and more interesting types of flows are possible if the equation of state is polytropic than if it is not. *J. L. Ericksen.*

Barua, S. N. Secondary flow in a rotating straight pipe. Proc. Roy. Soc. London. Ser. A. **227**, 133–139 (1954).

This paper considers the steady viscous flow in a rotating straight circular pipe. On the assumption that the flow remains the same in each section of the pipe, the problem is reduced to depend on two scalar functions. In the case of slow rotation the solutions are expanded in powers of a

Reynolds number defined by the angular velocity and the diameter of the pipe. To the first approximation it is shown that secondary motion has been set up, of which the projection of the stream lines on the cross-section of the pipe are calculated. The effect of rotation on drag is also studied.

Y. H. Kuo (Ithaca, N. Y.).

Ibrahim, Ali A. K., and Kabiel, Abdel Monem I. The theory of an oscillating cylinder viscometer. Z. Angew. Math. Physik 5, 398–408 (1954).

The authors have shown that their previous theoretical calculations on the viscous flow between coaxial oscillating cylinders [J. Appl. Phys. 23, 754–756 (1952)] are in agreement with experiments for the case of paraffin oil.

J. H. Kuo (Ithaca, N. Y.).

***Timman, R. Le potentiel vecteur et son application à l'analyse harmonique d'un écoulement à trois dimensions.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 351–361. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

In a work on the harmonic analysis of two-dimensional flows of a viscous incompressible fluid with adherence on the boundary, Kampé de Fériet [Quart. Appl. Math. 6, 1–13 (1948); these Rev. 9, 631] derived an integro-differential equation for the Fourier transform of the vorticity, the derivation making strong use of the stream function and of the property that its Laplacian is the vorticity. In the present paper the author obtains the appropriate extension to spatial flows by use, in place of the stream function, of a vector potential uniquely determined by the properties that its curl is the velocity, that it vanishes on the flow boundary, and has zero divergence in the interior. This vector potential has the essential property that its Laplacian is the vorticity vector, so that the details go through much as in the plane case. *D. Gilbarg* (Stanford, Calif.).

Yih, Chia-Shun. Stability of two-dimensional parallel flows for three-dimensional disturbances. Quart. Appl. Math. 12, 434–435 (1955).

In this short note, a relationship between the stability of two-dimensional parallel flows for three-dimensional disturbances and that for two-dimensional ones is obtained. Neither is the upper surface of the fluid necessarily assumed to be fixed, nor are the gravitational forces and variations in density and viscosity neglected. The author concludes that the primary flow is stable or unstable for a three-dimensional disturbance according as it is stable or unstable for a two-dimensional one at a lower Reynolds number, a milder slope, and a reduced pressure gradient. The laws of reduction for these three quantities are given. It is to be understood that these conclusions are based on a consideration of the linearized stability equations. The general approach used is attributed to C. C. Lin [Proc. Symposia Appl. Math., v. 5, McGraw-Hill, New York, 1954, pp. 1–18; these Rev. 16, 83]. *R. C. Di Prima.*

Mager, Artur. Three-dimensional laminar boundary layer with small cross-flow. J. Aeronaut. Sci. 21, 835–845 (1954).

This is a study of some boundary-layer flows in the category where the three-dimensional effects are small, so that the flows are perturbations of two-dimensional boundary-layer flows. The velocity components of the three-dimensional pattern are then given approximately by linear differential equations in which the plane-flow velocity components appear as variable coefficients. These are attacked by introducing a pair of stream functions, following Moore [NACA Tech. Note no. 2279 (1951); these Rev. 12, 872], expressing the external-flow variables in power-series form, and computing the stream functions in that form. This leads to a set of ordinary differential equations. The first-order cross-flow component w in the boundary layer is independent of the other two first-order components u and v. If w is independent of the cross-flow coordinate z, the converse is also true: u and w are unaffected, to first order, by the cross flow. This is a generalization of the result ("the independence principle") noted by the reviewer and others for the case when u, v, and w are all independent of z.

Some of the ordinary differential equations have been integrated numerically and the results plotted. These are applied to some examples: (1) a thin, spinning, cylindrical shell describing a circular path at small angle of yaw, (2) a flow, along a flat surface, simulating that in an S-shaped duct.

Next, a study is made of the effects of curvature of the surface in the cross-flow direction, again assuming small cross-flow velocities. An expansion is made in powers of a dimensionless curvature parameter, and the calculation is carried out for the linear term. The results are applied to an example involving a hyperbolically curved wall.

The reviewer calls attention to a similar, recent investigation by Wilkinson [Aeronaut. Quart. 5, 73–84 (1954); these Rev. 16, 83]. The conclusions do not seem to agree in all respects. *W. R. Sears* (Ithaca, N. Y).

v. Krzywoblocki, M. Z. On the development of the mathematical theory of the boundary layer. Comment. Math. Univ. St. Paul. 3, 51–66; corrections, 67 (1954).

Expository paper.

***Burgers, J. M. Some considerations on turbulent flow with shear.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 141–148. Academic Press Inc., New York, 1954. $9.00.

L'auteur se propose de calculer le moment de transfert turbulent de la vitesse dans la couche limite au voisinage d'une paroi plane, en utilisant un modèle hydrodynamique raisonnable. L'axe des y est normal à la paroi. La vitesse moyenne a pour composantes $(U, 0, 0)$ la vitesse d'agitation (u, v, w); U ne dépend que de y. Avec les variables de Lagrange, les équations choisies sont de la forme:

$$\frac{du}{dt}=-k(u-U)+Kv+F_u,$$

$$\frac{dv}{dt}=-kv-K(u-U)+F_v,$$

$$\frac{dw}{dt}=-kw+F_w,$$

où $k=\alpha_1\nu D^{-2}+\alpha_2 J^{1/2}D^{-1}$, $K=\alpha dU/dy$, α, α_1, α_2 sont des coefficients numériques. J est l'intensité de la turbulence. D est une longueur qui sert à corriger la forme trop linéaire de la loi de résistance. On peut alors calculer la valeur moyenne de $(u-U)v$, et en déduire une expression de la constante ydU/dy (coefficient de Kármán). On peut aussi trouver des solutions des équations initiales en variables d'Euler. De ces calculs, on déduit des expressions des coefficients α, α_1, α_2, qui en montrent la nécessité et en donnent une interprétation. *J. Bass* (Chaville).

Bers, Lipman. Existence and uniqueness of a subsonic flow past a given profile. Comm. Pure Appl. Math. **7**, 441–504 (1954).

This paper proves the existence and uniqueness of a steady two-dimensional potential subsonic flow of a perfect gas around a given profile. The profiles considered either have a sharp trailing edge (and are otherwise smooth), or they are entirely smooth. In the first instance (Problem 1) the velocity at infinity is prescribed, since the flow is then uniquely determined by the Kutta-Joukowsky condition. For smooth profiles both the velocity at infinity and the circulation are prescribed (Problem 2). One cannot, of course, expect in Problem 1 that a subsonic flow exists for arbitrary values of the free stream speed: what is shown is that there is a range of free stream speeds, $0<q_\infty<\hat{q}$, for which there exists a unique subsonic flow of the desired type, and that the maximum flow speed approaches sound speed as $q_\infty\to\hat{q}$. For Problem 2 the conclusion is similar. (Problem 1 was first considered by Frankl and Keldysh, who proved existence and uniqueness for sufficiently small free stream speeds. Insofar as Problem 2 is concerned, the present result slightly extends earlier work of Shiffman [J. Rational Mech. Anal. 1, 605–652 (1952); these Rev. 14, 510].)

In addition to the main results, there are theorems concerning possible stagnation points in the flow, the dependence of the flow on parameters, and the variation of flow speed as the conditions at infinity are changed. An interesting result of the last type is the following: a subsonic circulation-free flow past a profile symmetric with respect to the x-axis is symmetric, and the flow speed at all but two points of the profile is a strictly increasing function of q_∞. An alternate proof of existence, based on the Leray-Schauder theorem, is appended, and finally it is shown that a flow which is subsonic outside a circle must be uniform at infinity.

The main portion of the paper is, in reality, a general treatment of a homogeneous Neumann problem for a quasi-linear uniformly elliptic equation in an infinite region (the equation of potential flow is not uniformly elliptic, but a device noted by Shiffman allows the flow problem to be reduced to that case). The methods employed may be described broadly as being function-theoretic (as in the treatment of incompressible plane flow), but a whole arsenal of analysis comes into use. The theories of uniformization, quasi-conformal transformations, and pseudo-analytic functions, the Birkhoff-Kellogg-Schauder fixed-point theorem, and the classic existence theorems of Korn, Lichtenstein, and Schauder for elliptic differential equations, to mention the major tools, are all exploited. Also, in the proof of uniqueness the hodograph transformation of gas dynamics is used. These are formidable tools, and the paper is consequently not elementary; on the other hand, the problem attacked is very difficult and the results are exceptionally complete. *J. B. Serrin* (Minneapolis, Minn.).

Kočina, N. N. On a particular exact solution of the equations of unsteady one-dimensional motion of a gas. Doklady Akad. Nauk SSSR (N.S.) **97**, 407 (1954). (Russian)

Pressure $p(r,t)$, density $\rho(r,t)$, and $u(r,t)t^{-\nu}$ are expanded as series of the form $p(r,t)=\sum_{p=1}^{\infty} P_p(r/t)t^{-(p\nu(\gamma-1)+\nu)}$, where u is velocity, r the spatial coordinate, and $\nu=0, 1, 2$ for one-dimensional, cylindrically, or spherically symmetrical flow. The first coefficient functions of the p and ρ series are arbitrary, but all others are determined by recursion formulas. *J. H. Giese* (Havre de Grace, Md.).

Lidov, M. L. Exact solutions of the equations of one-dimensional unsteady motion of a gas, taking account of the forces of Newtonian attraction. Doklady Akad. Nauk SSSR (N.S.) **97**, 409–410 (1954). (Russian)

For an arbitrary initial density and special initial pressure distribution the author constructs for adiabatic exponent $\gamma=4/3$ a one-parameter family of spherically symmetrical motions of a gravitating gas. This includes an example of a periodically pulsating finite sphere of gas in a vacuum. For arbitrary γ he finds a class of motions that depends on the solutions of a non-linear second-order ordinary differential equation. *J. H. Giese* (Havre de Grace, Md.).

Žukov, A. I. On a family of exact solutions of the equations of hydrodynamics. Doklady Akad. Nauk SSSR (N.S.) **97**, 985–986 (1954). (Russian)

The author constructs solutions of the equations of steady axisymmetric inviscid compressible flow that involve an arbitrary function and solutions of a system of three first-order ordinary differential equations. *J. H. Giese.*

Sekerž-Zen'kovič, Ya. I. On the problem of flow with circulation about a circular cylinder for subsonic velocities. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 399–408 (1954). (Russian)

By the following adaptation of A. I. Nekrasov's method [same journal **8**, 249–266 (1944); these Rev. **7**, 38] the author finds the plane flow about the cylinder $x_1+iy_1=ae^{i\omega}$ with velocity V_∞, 0 at infinity. Let

$$\Phi(z,\theta)=x_1\partial\varphi/\partial x_1+y_1\partial\varphi/\partial y_1-\varphi(x_1,y_1)$$

be the Legendre transform of the velocity potential φ, where the velocity $-\partial\varphi/\partial x_1-i\partial\varphi/\partial y_1=Ve^{i\theta}=zV_\infty e^{i\theta}$. As in the Rayleigh-Janzen method, expand $\Phi(z,\theta)=\Phi_0+\tau_\infty^2\Phi_2+\cdots$, where $\tau_\infty=(\gamma-1)^{1/2}M_\infty[2+(\gamma-1)M_\infty^2]^{-1/2}$, substitute in the linear partial differential equation for Φ, and equate coefficients of τ_∞^{2n} to zero. In terms of $\zeta=1-\lambda^2-ze^{-i\theta}$ and its conjugate ζ_1, where $\lambda=\Gamma/4\pi aV_\infty$, the real-valued functions Φ_0 and Φ_2 satisfy Laplace's equation $\partial^2\Phi_0/\partial\xi\partial\xi_1=0$ and Poisson's equation

$$(*)\quad 2(\gamma-1)\frac{\partial^2\Phi_2}{\partial\xi\partial\xi_1}=(1-\lambda^2-\xi)^2\frac{\partial^2\Phi_0}{\partial\xi^2}+(1-\lambda^2-\xi_1)^2\frac{\partial^2\Phi_0}{\partial\xi_1^2}.$$

The mapping back onto the flow plane is accomplished by $(x_1+iy_1)V_\infty=2\partial\Phi/\partial\xi$. Then on the cylinder

$$(**)\qquad 2\partial\Phi/\partial\xi=\pm aV_\infty ie^{i\theta},$$

which must define the boundary values of $z(\tau_\infty,\theta)$ as a real function of θ to terms of order τ_∞^0 (τ_∞^2) for Φ_0 (Φ_2). Here Φ_0 is chosen to correspond to incompressible flow with circulation Γ, and Φ_2 must be such that $\partial\Phi_2/\partial\xi$ has only the singularities $(-i\lambda+\xi^{1/2})^{-1}$ (as for $\partial\Phi_0/\partial\xi$) or $\log(-i\lambda+\xi^{1/2})$, and $\partial\Phi_2/\partial\xi_1$ has the conjugate singularities. The author finds explicitly a solution Φ_2 of (*) with these properties, and from (**) the corresponding second approximation to the velocity distribution on the cylinder. *J. H. Giese.*

Longhorn, A. L. Subsonic compressible flow past bluff bodies. Aeronaut. Quart. **5**, 144–162 (1954).

Let (ζ,μ,ω) be ellipsoidal coordinates related to cartesian coordinates (x,y,z) by

$$x=K\mu\zeta,\quad y+iz=K(1-\mu^2)^{1/2}(\zeta^2-1)^{1/2}e^{i\omega},$$

where K is a constant. The author finds the first Rayleigh-Janzen approximation $\varphi=\varphi_0+M^2\varphi_1$ to the velocity-potential function for axisymmetric flow at Mach number M

about a prolate spheroid $\zeta=\zeta_0$, where φ_0 corresponds to incompressible flow. He expands φ as a series

$$\varphi=\sum_0^\infty g_{2n+1}(\zeta)P_{2n+1}(\mu)$$

of Legendre polynomials, where $g_{2n+1}(\zeta)$ are known functions involving integrals of Legendre polynomials and Legendre functions of the second kind. The approximate velocity on the ellipsoid is

$$\left[\frac{\partial\varphi}{\partial\mu}\right]_{\zeta_0}=\left[\frac{\partial\varphi_0}{\partial\mu}\right]_{\zeta_0}[1+M^2\sum_0^\infty P'_{2n+1}(\mu)H_{2n+1}(\zeta_0)],$$

where the M^2-term represents the effect of compressibility. For bluff ellipsoids for which the ratio of the length of equatorial to polar radius exceeds 0.4, three terms of the series are believed to suffice. H_1, H_3, and H_5 have been tabulated, and for large n the asymptotic form of H_{2n+1} has been determined. For comparative purposes the case of slender ellipsoids is also considered. *J. H. Giese.*

Woods, L. C. Compressible subsonic flow in two-dimensional channels with mixed boundary conditions. Quart. J. Mech. Appl. Math. **7**, 263–282 (1954).

This paper is concerned with the calculation of the subsonic potential flow of a perfect gas in a channel (i.e. a simply connected region of flow bounded by two streamlines, one of which can be at infinity) when the pressure is assigned over some contiguous section of the bounding streamlines (free boundary), while the shape of the remaining section is known. The author's method of solution is an approximate one, based on the assumption that ϕ and ψ are harmonic functions of the variables (r,θ), where $r=\int(1-M^2)^{1/2}d\log q^{-1}$, M=local Mach number, and ϕ, ψ, q, θ have the usual meanings; this assumption is good so long as $M^4-M_\infty{}^4\ll1$, and is exact for incompressible flow. He begins by deriving an integral equation for $r+i\theta$ as a function of (ϕ,ψ). In the physically important case when the assigned pressures are constant and the walls polygonal, the integral equation becomes a representation of $r+i\theta$. Four examples of this type are solved: the design of a bend in a channel; the flow of a stream up a step, with boundary layer separation; a jet impinging on a flat plate; Borda's mouthpiece. In a fifth example, flow of a jet deflected around a curved surface, the integral equation must be solved, the solution being effected by an iteration process which, at least in this case, converges rapidly. *J. B. Serrin* (Minneapolis, Minn.).

Ghaffari, Abolghassem. On some mathematical aspects of compressible flow. Pakistan J. Sci. **6**, 111–113 (1954).

Serini, Rocco. Adiabaticità nel movimento dei gas perfetti. Boll. Un. Mat. Ital (3) **9**, 241–243 (1954).

Syrovatskiĭ, S. I. Instability of tangential discontinuities in a compressible medium. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **27**, 121–123 (1954). (Russian)

Let the half-space $x>0$ (<0) be filled with fluid moving with constant velocity v_0 (0) parallel to the y-axis. By considering general exponential plane-wave solutions of the linearized system of partial differential equations for time-dependent perturbations subject to appropriate boundary conditions at the perturbed discontinuity surface, the author concludes that for all v_0 the motion is unstable. This contradicts L. D. Landau's earlier conclusion [C. R. (Doklady) Acad. Sci. URSS (N.S.) **44**, 139–141 (1944); these Rev. **6**, 191], obtained by use of plane waves of very special form, that for large enough v_0 the motion is stable.

J. H. Giese (Havre de Grace, Md.).

Chou, Pei Chi. Variational and Galerkin's methods in compressible fluid flow problems. J. Appl. Phys. **25**, 1551 (1954).

***Handbook of supersonic aerodynamics. Vol. 5. Section 15.** Navord Report 1488. U. S. Government Printing Office, Washington, D. C. $2.00.

The volume consists of tables of the thermodynamic and transport properties of atmospheric air, with explanations of the use of the tables and the methods used in their construction. They are based principally on calculations by Curtiss and Hirschfelder. These calculations take into account the known imperfections of the gas, and dissociation to chemical equilibrium. The viscosity table, and certain tables of virial coefficients, are based on a Lennard-Jones intermolecular potential. The results are given for dry air and for a range of absolute humidities. Almost all results given are in British engineering units. The pressure range covered is from 10^{-2} to 10^2 atmospheres, and the temperatures range from around 100°K to around 5000°K. The volume is concluded with short tables of isentropic changes and changes across shock fronts, based on the thermodynamic material presented earlier. *M. J. Lighthill.*

Thomas, T. Y. A remark on detached shocks. Proc. Nat. Acad. Sci. U. S. A. **40**, 1002–1004 (1954).

Let R be the radius of a sphere or circular cylinder in a supersonic flow without viscosity or thermal conductivity, let φ be the distance from the obstacle to the vertex of the shock, and let κ be the curvature of the shock at its vertex. By dimensional analysis and similarity arguments it is shown that φ/R and κR depend only on the free-stream Mach number and the ratio of specific heats. It should be remarked that these are merely special cases of the well known criterion for similarity of non-viscous flows about geometrically similar objects. *J. H. Giese.*

***Ludford, G. S. S. Two topics in one-dimensional gas dynamics.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 184–191. Academic Press Inc., New York, 1954. $9.00.

The author considers the one-dimensional unsteady flow of a gas with pressure-density relation $p=B\rho^\gamma$, where B and γ are constant. When $\gamma\neq1$ it is well known that this problem may be reduced to a second-order partial differential equation for the time t in terms of characteristic variables r, s [cf. Courant and Friedrichs, Supersonic flow and shock waves, Interscience, New York, 1948, p. 89, these Rev. **10**, 637]; when $\gamma=1$ a similar result holds, but the functional form of the equation is different. The author shows, however, that the equation for $\gamma=1$ is a limiting case of the equations for $\gamma\neq1$, and that the corresponding Riemann functions $v(P,Q;\gamma)$ depend continuously on γ in any finite range including $\gamma=1$, provided that $\overline{PQ}$ is suitably small. The second part of the paper determines the Riemann function for the differential equation satisfied by the space coordinate x as a function of r, s. *J. B. Serrin.*

Asaka, Saburô. On the velocity distribution over the surface of a symmetrical aerofoil at high speeds. I. Nat. Sci. Rep. Ochanomizu Univ. **4**, 213–226 (1954).

An outline is given of Imai's thin-wing-expansion method. The method is modified in order to simplify the procedure

for making higher-order approximations to the velocity distribution on aerofoil surfaces. Application of the theory to a symmetrical circular arc aerofoil is reserved for a later paper. *D. C. Pack* (Glasgow).

Lighthill, M. J. Oscillating airfoils at high Mach number. J. Aeronaut. Sci. **20**, 402–406 (1953).

Starting from Hayes' result [Quart. Appl. Math. **5**, 105–106 (1947); these Rev. **8**, 610] that, for sufficiently high Mach numbers, "any plane slab of fluid, initially perpendicular to the undisturbed flow, remains so as it is swept downstream and moves in its own plane under the laws of one-dimensional unsteady motion," the author obtains an approximation to the pressure on an airfoil in unsteady, hypersonic flow that retains terms through the cube of the ratio of airfoil (normal to free stream) to free stream sonic velocity and is subject to the restrictions that the absolute magnitude of this ratio be less than one. This result is applied to a symmetrical airfoil executing a pitching oscillation, and it is found that the results support those of Van Dyke [J. Aeronaut. Sci. **20**, 61 (1953)] but not those of Wylly [in the paper reviewed below], and also exhibit reasonable agreement with the corresponding results of Jones and Skan [Aeronaut. Res. Council, Rep. and Memo. **2749** (1953)] at $M=2$, which, according to the author, "augurs well for the present theory at the rather high Mach number for which it was designed to work." *J. W. Miles* (Los Angeles, Calif.).

Wylly, Alexander. A second-order solution for an oscillating two-dimensional supersonic airfoil. J. Aeronaut. Sci. **19**, 685–696, 704 (1952).

An iteration procedure is used to obtain the forces due to a pitching oscillation. The final results neglect terms of third order in (reduced) frequency and in amplitude of oscillation and thickness but indicate that the instability predicted by linearized theory [Possio, Pont. Acad. Sci. Acta **1**, 93–106 (1937); I. E. Garrick and S. I. Rubinow, NACA Tech. Note no. **1158** (1946)] for thickness ratios greater than $4\frac{1}{2}\%$ (for a modified double wedge section) will not be realized. Comparison with experiments [J. A. Beavan and D. W. Holder, J. Roy. Aeronaut. Soc. **54**, 545–586 (1950)] on a $7\frac{1}{2}\%$ biconvex section yields agreement that appears to be within the limitations of the data. Despite this agreement, the reviewer feels that the theoretical treatment is not entirely satisfactory. First, solutions are obtained as expansions in the neighborhood of the airfoil that evidently cannot be uniformly valid at large distances and therefore preclude the complete imposition of the appropriate initial (with reference to the streamwise coordinate) conditions. In the first-order solution this introduces no real difficulty, but since the first-order solution is not uniformly valid in the neighborhood of the leading-edge Mach waves and exhibits a singularity there, the second-order solution may exhibit a still more objectionable singularity [cf. M. J. Lighthill, Philos. Mag. (7) **40**, 1179–1201 (1949); these Rev. **11**, 518] and render impossible the imposition of the "initial" conditions. The author claims to circumvent this difficulty via an artifice previously used by Van Dyke [in the report reviewed below] and Courant and Friedrichs [Supersonic flow and shock waves, Interscience, New York, 1948, p. 365; these Rev. **10**, 637] that to some extent resembles Lighthill's extension (loc. cit.) of Poincaré's technique, but it is by no means clear that this has been accomplished since everything turns on whether the limiting results stated by the author are approached uniformly, and this question is not discussed. Finally, it should be noted that M. Van Dyke (private communication) has obtained results that differ from those of the present paper in both magnitude and sign of the second-order contribution to the damping in pitch of a wedge airfoil. Van Dyke also has extended Carrier's work on the oscillating wedge [J. Aeronaut. Sci. **16**, 150–152 (1949); these Rev. **10**, 493]. *J. W. Miles* (Los Angeles, Calif.).

Van Dyke, Milton D. A study of second-order supersonic flow theory. NACA Rep. no. **1081**, ii+23 pp. (1952).

The present paper, which supersedes NACA Tech. Note no. **2200** (1951), is based upon the writer's thesis [Calif. Inst. Tech., 1949]. Assuming isentropic flow (which is consistent with the "second-order" approximation), the resulting non-linear equation for the velocity potential is attacked by iteration, giving rise to the sequence

$$(\mathrm{A}_n)\qquad \Phi_{yy}^{(n)}+\Phi_{zz}^{(n)}-\Phi_{xx}^{(n)}=F_{n-1}(x,y,z)$$

together with the appropriate boundary conditions. Problem (A_1), for which $F_0=0$ and for the solution of which well-known techniques are available, leads to the usual "linearized" or "first-order" solution. The author obtains particular integrals of (A_2) for both plane and axially symmetric flow [cf. Van Dyke, J. Aeronaut. Sci. **18**, 161–178, 216 (1951); these Rev. **14**, 331] but only a partial particular integral for the general, three-dimensional problem. Second-order solutions obtained with these particular integrals are shown to be in excellent agreement with more exact solutions for particular examples. The second-order solution for a cone also is compared with Broderick's expansion through the fourth power (and including logarithms) of the cone angle [Quart. J. Mech. Appl. Math. **2**, 98–120 (1949); these Rev. **10**, 643] and is found to be superior thereto. As an example of a three-dimensional flow, the inclined cone is treated. In calculating the perturbation pressure, the author utilizes the exact (isentropic) expression for the pressure in terms of the perturbation velocity, stating that this is at least as convenient as the use of a consistent second-order approximation and may be more accurate. Particular attention is paid to the treatment of profile discontinuities in plane and axially symmetric flow and to their effect on the convergence of the iteration process, and it is also remarked that the iteration solution breaks down in the neighborhood of shock waves. (In this connection, reference is made to Lighthill's determination of the shock position in a conical flow [Philos. Mag. (7) **40**, 1202–1223 (1949); these Rev. **11**, 625]. The reviewer believes that Lighthill's more general paper on rendering iteration approximations uniformly valid [ibid. **40**, 1179–1201 (1949); these Rev. **11**, 518] also is pertinent to the present investigation.) A separate report [Van Dyke, NACA Tech. Note no. **2744** (1952); these Rev. **14**, 218] has been published on the numerical details of the second-order solution for a body of revolution. *J. W. Miles* (Los Angeles, Calif.).

Van Dyke, Milton D. Supersonic flow past oscillating airfoils including nonlinear thickness effects. NACA Tech. Note no. **2982**, 41 pp. (1953).

The author's summary reads: "A solution to second order in thickness is derived for harmonically oscillating two-dimensional airfoils in supersonic flow. For slow oscillations of an arbitrary profile, the result is found as a series including the third power of frequency. For arbitrary frequencies, the method of solution for any specific profile is indicated, and the explicit solution derived for a single wedge. Nonlinear

thickness effects are found generally to reduce the torsional damping, and so enlarge the range of Mach numbers within which torsional instability is possible. This destabilizing effect varies only slightly with frequency in the range involved in dynamic stability analysis, but may reverse to a stabilizing effect at high flutter frequencies. Comparison with a previous solution exact in thickness [Carrier, J. Aeronaut. Sci. **16**, 150–152 (1949); these Rev. **10**, 493; Van Dyke, Quart. Appl. Math. **11**, 360–363 (1953); these Rev. **15**, 177] suggests that nonlinear effects of higher than second order are practically negligible. The analysis utilizes a smoothing technique which replaces the actual problem by one involving no kinked streamlines [R. Courant and K. O. Friedrichs, Supersonic flow and shock waves, Interscience, New York, 1948, p. 399; these Rev. **10**, 637; Van Dyke, NACA Rep. **1081** reviewed above]. This strategem eliminates all consideration of shock waves from the analysis, yet yields the correct solution for problems which actually contain shock waves."

The previous treatment of second-order effects by A. Wylly [paper reviewed second above] is shown to be erroneous, while the semi-empirical approach of W. P. Jones [Aeronaut. Res. Council, Rep. and Memo. **2679** (1947)] is found to be valid only for pivot positions near mid-chord. The author's analysis includes only first-order terms in amplitude of oscillation (as befits a stability analysis), but a footnote indicates the differential equation governing the second-order effect in amplitude; the reviewer believes that the latter is incorrect due to the assumption of a complex, harmonic time dependence with the consequent implication that the end result is given by the real part of the complex solution (but this error does not affect any of the results given in the body of the paper). *J. W. Miles.*

Nonweiler, T. The wave drag of highly-swept wings—a comparison of linear theory and slender body theory. Coll. Aeronaut. Cranfield. Note no. **14**, i+14 pp. (1954).

The author compares his previous results [same Coll. Rep. no. **76** (1953); these Rev. **15**, 480] with those predicted by Ward's slender body theory [Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); these Rev. **10**, 644]. Disagreement is found only when the trailing edge is of finite thickness or the trailing edge is unswept and is attributed, respectively, to the quadratic terms in the pressure equation and to the reversal of the limits in which the aspect ratio and trailing edge angle limits tend to zero. The reviewer notes that the slender body and the linear theories satisfy the boundary condition of tangential flow at the surface of the wing and its plane of symmetry, respectively, and the latter (planar) approximation breaks down when the aspect ratio is no longer large compared with the thickness ratio.

J. W. Miles (Los Angeles, Calif.).

Gorgui, M. A. The effect of delta vanes on supersonic wings. Aeronaut. Quart. **5**, 251–279 (1954).

This is an application of cone-field theory to the problem of delta-shaped end-plates on a rectangular wing in supersonic flow. The analysis follows the lines of the particular version of cone-field theory given by Goldstein and Ward [same Quart. **2**, 39–84 (1950)]. It is shown that, depending on the thickness and aspect ratio of the end-plates, the gain in lift may on balance be more important than the increase of (wave) drag due to the presence of the plates. It would be interesting to compare the results obtained under supersonic conditions with the corresponding effect for low-speed flow. *A. Robinson* (Toronto, Ont.).

Kaplan, Carl. On transonic flow past a wave-shaped wall. NACA Rep. no. **1149** (1953), ii+12 pp. (1954).

Supersedes NACA Tech. Note no. **2748** (1952); these Rev. **14**, 425.

Payne, L. E. A note on a paper by Davies and Walters on "The effect of finite width of area on the rate of evaporation into a turbulent atmosphere." Quart. J. Mech. Appl. Math. **7**, 283–286 (1954).

L'auteur remplace la solution obtenue par Davies et Walters [même J. **4**, 466–480 (1951); ces Rev. **13**, 657] par une expression plus adaptée aux calculs numériques. L'auteur montre, en outre, que l'équation aux dérivées partielles admet une solution compatible avec les conditions aux limites seulement lorsque la constante m qui rentre dans cette équation, satisfait aux inégalités $0<m<1$.

M. Kiveliovitch (Paris).

Kao, Shih-Kung. Harmonic wave solutions of the nonlinear vorticity equation for a rotating viscous fluid. J. Meteorol. **11**, 373–379 (1954).

In this paper the author obtains solutions of the nonlinear vorticity equation where the effect of friction, as represented by the Navier-Stokes terms, is included. The fluid is assumed to be incompressible and all vertical velocities are neglected. The author treats two cases: firstly plane flow, where the variation of the Coriolis parameter with latitude is assumed to be constant; and, secondly, flow on a rotating sphere. For the case of plane flow the solutions give waves which can either be damped, amplified, or remain stationary, depending on the amplitude, wave-length and the vertical profile of the velocity components. It is found that when the vertical profile of the velocity components is a sinusoidal function of z, where z is the elevation, proportional to z, or independent of z, the waves are always damped; but when the profile is a combination of hyperbolic sines and cosines of z, all three types of development are possible. However, the reviewer feels that these results would be considerably modified by the inclusion of vertical velocities.

For the case of flow on a rotating sphere, harmonic solutions are again obtained in which damped or amplified wave patterns are possible. The vertical dependence enters through the factor $r^{\frac{1}{2}}Z_{n+\frac{1}{2}}(s^{\frac{1}{2}}r)$, where r is radial distance from the center of the earth, Z is a Bessel function of order $n+\frac{1}{2}$, and the velocity components contain a factor $e^{-\nu st}$, ν being the coefficient of viscosity. It is found on putting $s=0$, that the flow reduces to a solid rotation relative to the rotating earth so that no steady wave-pattern can exist in this model; however, the author demonstrates the existence of waves with zero wave-velocity and changing wave-amplitude.

M. H. Rogers (Urbana, Ill.).

Goldsbrough, G. R. Wind effects on the motion of the sea in an infinite channel and in a rectangular gulf. Proc. Roy. Soc. London. Ser. A. **222**, 477–489 (1954).

Déjà en 1905 Ekman [Ark. Mat. Astr. Fys. **2**, no. 11 (1905)] a étudié le problème en introduisant la rotation de la terre. C'est en 1923 que Jeffreys [Philos. Mag. (6) **46**, 114–125 (1923)] a étudié le problème en introduisant l'effet du frottement. L'auteur reprend la question et étudie dans la première partie l'effet d'un vent stationnaire sur le mouvement ainsi que la déformation de la surface d'une mer ayant la forme d'un canal illimité à rives parallèles. En utilisant les conditions aux limites classiques, c'est-à-dire mouvement nul aux extrémités, l'auteur obtient, en sup-

posant la rotation angulaire et le coefficient de viscosité turbulente constantes, deux types de solutions suivant que le mouvement stationnaire dépend ou ne dépend pas de y (la dimension parallèle aux rives). Dans la deuxième partie l'auteur étudie le cas d'une mer limitée à une extrémité, genre golfe. L'auteur obtient une solution en changeant, un peu artificiellement, les conditions aux limites et en utilisant les deux types de solutions obtenues précédemment.

M. Kiveliovitch (Paris).

Kolmogorov, A. N. On a new variant of M. A. Velikanov's gravitational theory of motion of suspended sediment. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **9**, no. 3, 41–45 (1954). (Russian)

L'auteur étudie en détails le travail de M. A. Velikanoff [même Vestnik **8**, no. 8, 45–55 (1953)], et montre que les résultats obtenus ne correspondent pas à la réalité du phénomène. L'erreur provient de la séparation complète des bilans énergétiques de la phase liquide et de la phase solide.

M. Kiveliovitch (Paris).

Carrier, G. F. On acoustic resistance to the transient motions of an immersed shell. Rev. Fac. Sci. Univ. Istanbul (A) **19**, 8–12 (1954).

The pressure distribution on the surface of a shell (plane, circular cylinder, or sphere) associated with the acoustic radiation generated by the motion of the shell is investigated. Several elementary results are obtained and are used to deduce an appropriate "Green's function" by which the surface pressure on a cylindrical shell due to an arbitrary shell motion may be represented. (From the author's summary.)

L. M. Milne-Thomson (Greenwich).

Elasticity, Plasticity

Green, A. E., and Wilkes, E. W. Finite plane strain for orthotropic bodies. J. Rational Mech. Anal. **3**, 713–723 (1954).

This work is based on the usual theory of anisotropic elastic bodies with strain energy W. Such bodies are called orthotropic if they possess three orthogonal planes $x_1=0$, $x_2=0$, and $x_3=0$, which are planes of elastic symmetry at each point in the undeformed state. It is shown for orthotropic bodies that W reduces to a function of e_{11}, e_{22}, e_{33}, e^2_{12}, e^2_{23}, e^2_{31}, $e_{12}e_{23}e_{31}$, where e_{ik} is the Green-St. Venant strain tensor.

The case of plain strain in $x_3=0$, superposed on a uniform extension in the x_3-direction, is considered in detail. Assuming equilibrium and the absence of body forces, Airy's stress function ϕ can be introduced. Then the stress-strain relations and the equilibrium conditions yield a system of three partial differential equations for $\phi(\theta_1, \theta_2)$, $x_1(\theta_1, \theta_2)$, and $x_2(\theta_1, \theta_2)$, where θ_α and x_α are the coordinates of a point in the deformed and in the undeformed state, respectively. This system is basic for plane strain of orthotropic bodies. As an example, the problem of flexure of a cuboid is partly solved for arbitrary W and compressible materials. It is solved completely for incompressible materials.

Finally, the authors write down the special form of the basic system mentioned above in the special case of materials which are transversely isotropic (symmetric with respect to any plane containing the x_3-axis). As an example, the inflation and extension of a hollow cylinder is partly solved for compressible bodies. This generalizes the solution for incompressible bodies which was given by Ericksen and Rivlin [same J. **3**, 281–301 (1954); these Rev. **16**, 88].

W. Noll (Berlin).

Hearmon, R. F. S. 'Third-order' elastic coefficients. Acta Cryst. **6**, 331–340 (1953).

The author's object is to determine the forms of the cubic terms in the strain energy of a perfectly elastic body appropriate to the various classes of crystals. He uses explicit calculation and obtains results in agreement with those of Fumi [Physical Rev. (2) **83**, 1274–1275 (1951)]. For the isotropic case there are at most three independent coefficients, as has long been known. [The author in discussing an erroneous reduction of the three to two is apparently unaware of the general method which enables one to determine at once the form of the strain energy in the isotropic case [e.g. the reviewer, §8 and §43 of J. Rational Mech. Anal. **1**, 125–300 (1952); **2**, 595–616 (1953); these Rev. **13**, 794; **15**, 178]. Since the author's paper appeared, this classical method has been generalized to the anisotropic case in §6–§7 of Ericksen and Rivlin, ibid. **3**, 281–301 (1954); these Rev. **16**, 88. While Ericksen and Rivlin work out the details only for the case of transverse isotropy, their method is entirely general, and not only does it avoid the necessity for lengthy calculation but also it applies to strain energies of any form.]

C. Truesdell (Bloomington, Ind.).

Manacorda, Tristano. Sopra un principio variazionale di E. Reissner per la statica dei mezzi continui. Boll. Un. Mat. Ital. (3) **9**, 154–159 (1954).

E. Reissner [J. Math. Physics **29**, 90–95 (1950); these Rev. **12**, 301] has obtained for classical linear elasticity an elegant variational principle which is equivalent to the differential equations of equilibrium, the stress-strain relations, and mixed boundary conditions. Subsequently [ibid. **32**, 129–135 (1953); these Rev. **15**, 369] he has generalized it to a particular class of theories of finite strain. Apparently unaware of the second paper of Reissner, the author attempts to obtain an extension to the general theory of finite elastic strain. [How far this attempt can be successful may be seen in advance by noticing that in Reissner's original proof the theory of elasticity is defined by (1) strain energy = function of stress, and (2) strain = gradient of energy with respect to stress, but (3) makes no direct use of the symmetry of the stress tensor, which emerges only through the stress-strain relations. All that is needed is to find a formalism for the finite theory in which (1) and (2) hold, then duplicate Reissner's steps. As is well known, (1) is not true in general (i.e. the stress-strain relations cannot be inverted in the large), but if (1) is assumed (2) follows, provided the "stress" is taken as Kirchhoff's double vector $T^{i\alpha}$, which is not symmetric [cf. (3) and §39 of the reviewer's paper, J. Rational Mech. Anal. **1**, 125–300 (1952); these Rev. **13**, 794]. The author follows this plan, although he fails to mention that (1) is a restriction which generally will invalidate the variational principle except for sufficiently small portions of the material. It should be noted also that the variational integrand as written is not a scalar unless both deformed and undeformed systems of co-ordinates are rectangular Cartesian.] The author concludes by using the variational principle to derive the stress-strain relations of Finger for isotropic elastic bodies.

C. Truesdell.

Gulkanyan, N. O. On the torsion of prismatic bars with a rectangular normal section in the presence of longitudinal cracks. Akad. Nauk Armyan. SSR. Izvestiya Fiz.-Mat. Estest. Tehn. Nauki 5, no. 2, 67–96 (1952). (Russian. Armenian summary)

Aleksandryan, E. A., and Gulkanyan, N. O. Torsion of bars with cross-sections in the form of a channel and a T. Akad. Nauk Armyan. SSR. Izvestiya Fiz.-Mat. Estest. Tehn. Nauki 6, no. 3, 37–51 (1953). (Russian. Armenian summary)

In the first paper the exact solutions of Saint-Venant's torsion problem are given for three types of rectangular beams with longitudinal slits. These solutions are obtained with the aid of a special device, in the form of the infinite series of particular solutions of Prandtl's equation. The following cases are considered (a) beam with one longitudinal crack (b) two longitudinal cracks at the midpoints of the opposite sides of the beam (c) an internal longitudinal crack, symmetrically placed with respect to the center of the beam. It is shown that the infinite systems of algebraic equations arising in the determination of coefficients in the series are completely regular.

Similar techniques are employed in the second paper to solve the torsion problems for polygonal beams with the channel (U) and T-cross sections. The paper is accompanied by tables of torsional rigidities for several ratios of the web and flange dimensions. *I. S. Sokolnikoff.*

*Reissner, H., and Reissner, E. **Torsion of a circular cylindrical body by means of tractions exerted upon the cylindrical boundary.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 262–273. Academic Press Inc., New York, 1954. $9.00.

Starting with the basic differential equations for axially symmetric deformations, the problem of cylindrical bodies having curvilinear aeolotropy and twisted by means of tractions over the boundaries is first solved by means of Fourier series in the manner originally given by Filon [Philos. Trans. Roy. Soc. London. Ser. A. 198, 147–233 (1902)] for isotropic cylinders. Certain special cases are treated and the range of validity of the results is determined when approximations of the Bessel functions involved in the general solution are used. It is also shown how, without the use of Fourier series, the elementary beam theory for torsion with distributed loads may be modified to give a result which is in agreement with the original exact result when the ratio of the radius of the beam to its length is not too large. The paper concludes with the solution for the effect of a solid core on the twisting of a thin cylindrical shell. *R. M. Morris* (Cardiff).

Chandra Das, Sisir. On the concentration of stresses due to a small elliptic inclusion on the neutral axis of a deep beam under constant bending moment. Z. Angew. Math. Physik 5, 389–398 (1954).

Sveklo, V. A. Lamb's problem for mixed boundary conditions. Doklady Akad. Nauk SSSR (N.S.) 95, 737–739 (1954). (Russian)

W. J. Smirnoff et S. L. Soboleff ont été les premiers à donner une solution exacte du problème dans le cas où les conditions aux limites sont indépendantes des tensions. L'auteur donne une solution complète dans le cas où les conditions aux limites sont mixtes et trouve comme solution limite la solution connue du problème de Lamb mais où l'impulsion tangentielle est différente de zéro. *M. Kiveliovitch* (Paris).

Žgenti, V. S. Application of functional analysis to a sloping elastic shell having the form of an elliptic paraboloid. Doklady Akad. Nauk SSSR (N.S.) 90, 9–11 (1953). (Russian)

The author employs an inequality of K. O. Friedrichs [Ann. of Math. (2) 48, 441–471 (1947); these Rev. 9, 255] and a theorem of S. L. Sobolev [Some applications of functional analysis to mathematical physics, Izdat. Leningrad. Gos. Univ., 1950; these Rev. 14, 565], in order to show the uniform convergence of a Ritz sequence [see S. G. Mihlin, Direct methods in mathematical physics, Gostehizdat, Moscow, 1950; these Rev. 16, 41] to the solution of a boundary-value problem occurring in the theory of elastic shells. *J. B. Diaz* (College Park, Md.).

Hart, V. G. Equilibrium of membranes elastically supported at the edges. Quart. Appl. Math. 12, 408–412 (1955).

The author considers the problem of finding the statical deflection of a membrane originally lying in the neutral plane, when subjected to a uniform pressure on one side, the edges being elastically supported. This means that the edge can move in a direction perpendicular to the neutral plane, but is restrained at any point by a force proportional to the deflection at that point. Small deflections only being considered, the tension is a constant and the problem reduces to solving a boundary-value problem for an edge of arbitrary shape. Solutions are given for a membrane in the form of an equilateral triangle and a rectangular membrane. *R. Gran Olsson* (Trondheim).

Serafimov, Petar R. Beitrag zur Theorie der Kegel- und Zylinderschalen. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire 6 (1953), no. 6, 27 pp. (1954). (Macedonian. German summary)

The author derives in Ch. I the results of F. Tölke [Ing.-Arch. 9, 282–288 (1938)]. In Ch. II and III Tölke's results are applied to conical and cylindrical shells with constant thickness and rotationally symmetrical loading and known results derived. In Ch. IV the same method is applied to conical shells with linearly varying thickness and rotationally symmetrical loading and thus the solution of the problem obtained [cf. Nollau, Z. Angew. Math. Mech. 24, 10–34 (1944)].

Finally, in Ch. V, the author declares that, in the same way as in the previous case, the cylindrical shell with linearly varying thickness and rotationally symmetrical loading can be treated and similar results obtained. But the suggestions given by the author in this respect are insufficient, as $\cos\alpha$ and the term with $\cos\alpha$ vanish for $\alpha=\pi/2$. *T. P. Anđelić* (Belgrade).

Olszak, Wacław. Généralisation de l'analogie de la membrane élastique aux problèmes des systèmes anisotropes. Arch. Mech. Stos. 5, 89–106 (1953). (Polish. French summary)

The Prandtl analogy of the uniformly stressed membrane subjected to a normal load with an isotropic prismatic bar subjected to torsion is described in detail by R. D. Mindlin and M. G. Salvadori [Hetényi (ed.), Handbook of experimental stress analysis, Wiley, New York, 1950, pp. 700–827]. The analogy consists in the fact that the partial differ-

ential equation for the displacement of the membrane is identical with the stress function differential equation of an isotropic bar, if the displacements of the membrane are small. The differential equations of anisotropic bars differ from that of a membrane, but the author shows that by a suitable transformation of the coordinates they can be reduced to the isotropic case, and the membrane analogy may also be conveniently applied. By using the membrane analogy the author demonstrates that the cross-sections of a circular anisotropic bar subjected to torsion must be warped, finds the conditions for which the cross-sections of an elliptic anisotropic bar will remain plain, and discusses other seemingly difficult anisotropic bars which yield easy solutions by membrane analogy. *T. Leser.*

Stippes, M., and Beckett, R. E. Symmetrically loaded circular plates. J. Franklin Inst. **257**, 465–479 (1954).

Calculation of the large transverse deflections of thin circular elastic plates, when the applied load and edge support are radially symmetric, involves the solution of a system of two non-linear, simultaneous, ordinary differential equations for the stress function and the transverse deflection as functions of the radial distance ρ. This system, due to von Kármán [Encyc. Math. Wiss., Bd IV, Teilbd 4, Teubner, Leipzig, 1910, p. 348] only admits of an exact solution in simple form in exceptional cases; and in general only an approximate solution may be found after much computation. The main object of the present paper is to present an analysis whereby the computation is systematized and thereby lessened. Attention is confined to the cases when the plate edge is either built-in or simply supported, and when either the edge radial membrane stress or the edge radial displacement is zero. Certain Green's functions for the equation $\rho\nabla_\rho^4F=0$ appropriate to the various boundary conditions are used to reformulate the problem in terms of a non-linear integral equation. The subsequent treatment requires that the applied load is such that the plate slope, on the basis of linear (i.e. bending) theory is expanded either as a Fourier-Bessel or as a Dini series according as the plate edge is either built-in or simply supported, respectively. In the more general non-linear case the plate slope is correspondingly expressed in terms of either a series of Bessel functions or a Dini series of Bessel functions. The coefficients in these series satisfy an infinite set of simultaneous cubic equations which in general only admits of approximate computational treatment. Bessel functions of the first kind and of order one only occur. No specific problems are completely solved, and the latter part of the paper is devoted to the discussion of the evaluation of certain fundamental integrals and of approximations to the solution. *H. G. Hopkins* (Fort Halstead).

***Federhofer, Karl. Der senkrecht zu seiner Ebene belastete, elastisch gebettete Kreisträger.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 242–250. Academic Press Inc., New York, 1954. $9.00.

Setzt man voraus, dass der Kreisträger auf einer Unterlage gebettet sei, die jeder Verschiebung v einen elastischen Widerstand c_1v (Winklersche Annahme mit c_1 als Bettungsziffer), und jeder Verdrehung β des Querschnittes ein Moment $c_2\beta$ ("elastische Drehbettung") entgegensetze, so ist die für dieses statische Problem massgebende Differentialgleichung für v ein Sonderfall jener Gleichung die Verf. über die Kippsicherheit des kreisförmig gekrümmten Trägers bei gleichmässiger Randbelastung entwickelt hat [Österreich. Ing.-Arch. **4**, 27–44 (1950); diese Rev. **11**, 702]. Bedeutet a den Halbmesser der Schwerachse des Kreisringes, so lautet diese Gleichung für die homogene Verschiebung $y=v/a$:

$$y^{\mathrm{viii}}+A_2y^{\mathrm{vi}}+A_4y^{\mathrm{iv}}+A_6y^{\mathrm{ii}}+A_8y=0,$$

wo alle Ableitungen von y nach der Winkelkoordinate φ des Bogens zu nehmen sind. Die Konstanten A_2 bis A_8 ergeben sich als abhängig von der Radialbelastung q des Trägers, der Form des Querschnitts und den Werten des elastischen Widerstandes c_1 und c_2. Die als Fundamente gebauten Kreisringträger haben rechteckigen Querschnitt, bei denen die Kennzahl q_s für den Einfluss der Querschnitteverwölbung sehr klein ausfällt und daher gleich Null gesetzt werden kann [Federhofer, Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa **157**, 299–320 (1950); diese Rev. **12**, 143]. Ausserdem fällt beim Rechteckquerschnitt der Schubmittelpunkt in den Schwerpunkt, sodass die reduzierte Koordinate q_z des Schubmittelpunktes gleich Null wird. Da alle Konstanten A_2 bis A_8 linear von $1/(\rho_s-\rho_z^2)$ abhängen, reduziert sich die vorstehende Gleichung auf eine Gleichung sechster Ordnung von der Form

$$y^{\mathrm{vi}}+a_2y^{\mathrm{iv}}+a_4y^{\mathrm{ii}}+a_6y=0,$$

wobei die Konstanten a_2 bis a_4 durch das Verhältnis der Biegungs-zur Drillungssteifigkeit EJ_z/GJ_D und die dimensionlosen Grössen $\bar{c}_1$, $\bar{c}_2$ mit den Bettungsgrössen c_1, c_2 durch $\bar{c}_1=c_1a^4/EJ_z$, $\bar{c}_2=c_2a^2/EJ_z$ bestimmt sind. Bei Vernachlässigung der Wirkung der elastischen Drehbettung ($c_2=0$) folgt

$$y^{\mathrm{vi}}+2y^{\mathrm{iv}}+(1+\bar{c}_1)y^{\mathrm{ii}}-\bar{c}_1y=0,$$

eine Gleichung die sich unmittelbar aus der vom Verf. aufgestellten Grundgleichung für die Berechnung des senkrecht zu seiner Ebene belasteten Bogenträgers ergibt [vgl. Federhofer, Z. Math. Phys. **62**, 40–63 (1913)]. *R. Gran Olsson* (Trondheim).

Paria, Gunadhar. Stresses in an infinite strip due to a nucleus of thermo-elastic strain inside it. Bull. Calcutta Math. Soc. **45**, 83–87 (1953).

An infinite-strip thin plate of isotropic elastic material is bounded by the coordinate lines $x=\pm a$ in any plane parallel to the faces. The distribution throughout the plate of the averaged stresses is to be determined when a nucleus of thermo-elastic strain is acting at the origin while the edges of the strip are kept stress-free. The nucleus is such as would be produced if a prismatic plate volume-element (having an area element as plane projection) is heated to a certain temperature while the remaining part of the plate is held at zero temperature. The averaged stress distribution due to a nucleus in an infinite plate was given by J. N. Goodier [Philos. Mag. (7) **23**, 1017–1032 (1937)]; the stresses on the edges of the strip produced by the nucleus are then eliminated by the superposition of a regular solution. The superposed stresses are determined by means of the stress functions of B. Sen [ibid. (7) **26**, 98–119 (1938)] and are expressed ultimately as Fourier integrals. An approximate formula is then derived, using two terms of the cosine series, for the maximum normal stress in the vicinity of the points $x=\pm a$ and $y=0$. *W. Nachbar.*

Litvinov, M. V. Solution of the problem of the theory of elasticity for an infinite strip by the method of finite differences. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 117–121 (1953). (Ukrainian. Russian summary)

The author shows that the "influence numbers" (inverse matrix) of the infinite system of linear equations of the

problem mentioned in the title may be obtained to any degree of accuracy by means of Gauss' algorithm.

J. B. Diaz (College Park, Md.).

Dlugač, M. I. Solution of mixed problems of the theory of elasticity by the method of grids. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 451–455 (1953). (Ukrainian. Russian summary)

In this paper the fundamental formulas of strength of materials for determining strains in systems of bars are extended in order to obtain formulas for the strains in the plane problem of the theory of elasticity, when this problem is attacked by the method of finite differences. (Free translation from the author's summary.) *J. B. Diaz.*

Tesch, H. Die Ausbildung eines Elastizitäts-Störmoments beim schwingenden Pendel mit elastischer Schneide bzw. elastischer Unterlage. Z. Angew. Math. Mech. **34**, 391–404 (1954). (English, French and Russian summaries)

Foulkes, J. The minimum-weight design of structural frames. Proc. Roy. Soc. London. Ser. A. **223**, 482–494 (1954).

Consider a structural frame which derives its strength from the plastic bending action and carries given static loads. The problem considered in the paper may be stated as follows: what cross-sectional dimensions must the members be given in order that the loads may be sustained and the weight of the frame be minimum? The author shows that under certain conditions this problem may be reduced to minimizing a linear function whose variables are constrained by linear inequalities. [This result and some of its consequences were discussed in a previous paper by the author, Quart. Appl. Math. **10**, 347–358 (1953).]

A geometrical interpretation of this problem enables the author to establish two extremum principles by the use of which upper and lower bounds for the minimum weight could be found. The paper concludes with an application of these extremum principles to the design of a simple practical structure. *E. T. Onat* (Ankara).

Link, H. Über den geraden Knickstab mit begrenzter Durchbiegung. Ing.-Arch. **22**, 237–250 (1954).

Die Untersuchung mit Hilfe der elementaren Biegetheorie eines geraden Knickstabes, der durch ebene, starre Flächen in seiner Durchbiegung begrenzt wird, führt auf eine bisher unbekannte Durchschlagerscheinung. Unabhängig von der Art der Lagerung der Stabenden stützt sich nach Überschreitung der zugehörigen Eulerschen Knicklast der Stab an der Stelle seiner grössten Durchbiegung gegen eine der Stützflächen ab. Die sich dabei ausbildende Stützkraft wächst mit zunehmender Stablast an, während sich gleichzeitig die Stabkrümmung an der Anlagestelle verringt. Erreicht die Stabkrümmung an der Anlagestelle die Krümmung der Stützfläche, bildet sich bei weiterer Steigerung der Stablast ein an der Stützfläche anliegendes, ungekrümmtes daher momentenfreies Stabstück aus, an dessen Enden Einzelkräfte von der Stützfläche auf den Stab übertragen werden. Die Länge dieses Stabteiles wächst mit der Stablast solange an, bis diese die Grösse der Eulerschen Knicklast des als an beiden Enden eingespannt anzusehenden geraden Stabteiles erreicht. Der nun einsetzende Knickvorgang des bis dahin an der Stützfläche anliegenden Stabteiles kann wegen der zunehmenden Knicklänge mit sich verringernder Stablast aufrecht erhalten werden. Bei unveränderter Stablast schlägt daher der mittlere Stabteil durch und kommt an der gegenüberliegenden Stützfläche zur Anlage. Bei weiterer Steigerung der Stablast kann der Stab immer als aus Teilstäben zusammengesetzt aufgefasst werden, deren Lagerung den hier untersuchten drei Grundfällen entspricht. Nach Ansicht des Verf. ist es daher nicht zu erwarten, dass eine Weiterführung der Untersuchung noch zu grundsätzlich neuen Erkenntnissen führt.

R. Gran Olsson (Trondheim).

Svešnikov, A. G. On the uniqueness of solution of exterior problems of the theory of steady elastic vibrations. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 253–256 (1954). (Russian)

In a previous paper on the same problem [same journal **17**, 443–454 (1953); these Rev. **15**, 268], the author based the justification of his "principle of limiting absorption" on a lemma (p. 448 of the paper) which, as was called to his attention by I. N. Vekua and V. D. Kupradze, is not true. In the present paper the author employs a different method to justify the "principle of limiting absorption".

J. B. Diaz (College Park, Md.).

Stanišić, Milomir M. Free vibration of a rectangular plate with damping considered. Quart. Appl. Math. **12**, 361–367 (1955).

It is shown that the modes of vibration of uniform plates having uniformly distributed viscous damping are unaffected by the damping. Frequencies for three modes of a square clamped plate with damping are computed with Galerkin's method and shown to be in agreement with earlier results by D. Young [J. Appl. Mech. **17**, 448–453 (1950)].

S. Levy (Washington, D. C.).

Morrison, J. A. Closure waves in helical compression springs with inelastic coil impact. Quart. Appl. Math. **11**, 457–471 (1954).

A helical compression spring, fixed at one end, is subjected to impact on its free end. A theory describing post-impact behavior of such springs, taking into account closure of individual coils, has been given by E. H. Lee [Proc. Symposia Appl. Math., v. 5, McGraw-Hill, New York, 1954, pp. 123–136; these Rev. **15**, 911]. The present paper describes briefly the portion of this theory in which inelastic collisions between adjacent coils are assumed. Application is then made to the following cases: (a) a constant or a uniformly retarded velocity is applied impulsively to the free end; (b) a mass attached to the free end is given a velocity impulsively. Time histories of closure waves and spring stresses are discussed in detail for each case.

W. Nachbar (Seattle, Wash.).

Musgrave, M. J. P. On the propagation of elastic waves in aeolotropic media. I. General principles. Proc. Roy. Soc. London. Ser. A. **226**, 339–355 (1954).

The author gives a derivation of equations governing wave motion in an anisotropic elastic medium, assuming linear elasticity, which seems to this reviewer to be less satisfactory than that which treats these as surfaces of discontinuity and is given, e.g., in Love, "A treatise on the mathematical theory of elasticity" [4th ed., Cambridge, 1927]. His main contribution seems to be a geometric construction for wave surfaces. *J. L. Ericksen.*

Musgrave, M. J. P. On the propagation of elastic waves in aeolotropic media. II. Media of hexagonal symmetry. Proc. Roy. Soc. London. Ser. A. **226**, 356–366 (1954).

Results given in the first paper of this series are used to obtain information concerning wave surfaces and related surfaces for transversely isotropic materials. Relevant numerical data and graphs of sections of these surfaces are given for zinc and beryl. *J. L. Ericksen.*

***Velasco de Pando, Manuel. Plasticidad (nueva teoria y aplicaciones). [Plasticity (new theory and applications).]** Patronato de Publicaciones de la Escuela Especial de Ingenieros Industriales, Madrid, 1954. xvi+256 pp. 130 pesetas.

Chapter headings: 1. Preliminary notions. 2. General principles of the new theory. 3. Study of the new plasticity by means of absolute strains. 4. Elastic-plastic bending of beams. 5. Elastic-plastic torsion. 6. Cylindrical shells. 7. Spherical shells in a plastic state. 8. Plane strain. 9. Classical plasticity as a special case of our theory of plasticity. 10. First approximation in the new theory. 11. Plastic mass compressed between two parallel planes. 12. Solids of revolution with loads independent of the meridian. 13. Torsion of cylinders of arbitrary cross-section. 14. Application of the theories of ch. 11 to a stretched rectangular specimen. 15. Study of the resulting equation for tangential stress in cases of plasticity in two dimensions. *Table of contents.*

Green, A. P. A theory of the plastic yielding due to bending of cantilevers and fixed-ended beams. I. J. Mech. Phys. Solids **3**, 1–15 (1954). Read the note on p. 1337

The author first investigates a uniformly tapering truncated cantilever subjected to a concentrated shear force at the end, under conditions of plane strain. A plastic-rigid material is assumed and velocity fields are found for all values of the parameters except for very short beams. The author then solves the following related problems: uniform cantilever with various weak supports, uniform cantilever under combined axial and shearing loads, and tapering cantilever under conditions of plane stress.

According to limit-analysis theorems, the load determined in each case is an upper bound on the yield-point load. The author acknowledges this but claims it is "unlikely" that the yield-point loads are overestimated. In only one special case a stress field is constructed to verify this conclusion. The reviewer doubts its validity for long cantilevers with little taper, since elementary beam-theory limit analysis furnishes slightly lower upper bounds.

The particular case of a sheared uniform cantilever was previously treated by Onat and Shield [Brown Univ. Tech. Rep. A11-103 (1953)] although the author is apparently unaware of this paper. *P. G. Hodge, Jr.*

Gotusso, Guido. Sul comportamento dei continui al di là del dominio elastico. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **17**(86), 384–406 (1953).

This paper discusses the derivation of equations governing the deformation of continua beyond the elastic range. Various types of mechanical behaviour are considered and attention is directed especially towards the inclusion of time effects, but essentially the material presented is standard. *H. G. Hopkins* (Fort Halstead).

Hill, R. On Inoue's hydrodynamical analogy for the state of stress in a plastic solid. J. Mech. Phys. Solids **2**, 110–116 (1954).

In recent papers Inoue [J. Phys. Soc. Japan **7**, 119–121, 518–523 (1952); these Rev. **14**, 931; J. Aeronaut. Sci. **19**, 783–784 (1952); Doshisha Engrg. Review **1952**, Special Paper No. 1] has pointed out an analogy between the steady flow of a compressible fluid and equilibrium states of stress in a plastic solid. While Inoue's work is restricted to two-dimensional problems (plane or axially symmetric), the present paper begins with the discussion of the three-dimensional analogy. The state of stress at a generic point of the plastic solid is found to consist of a state of hydrostatic pressure and a uniaxial state of stress. To within an arbitrary additive constant, the pressure in the plastic solid equals that in the fluid, while the uniaxial stress has the direction of the velocity in the fluid, its intensity being the product of the fluid density and the square of the velocity. The yield condition of the plastic solid depends on the equation of state of the barotropic fluid. The author points out the formal character of this analogy which ignores the flow rule of the plastic solid: non-trivial three-dimensional plastic states can be obtained by this analogy only if the plastic potential is singular. The converse transformation from the plastic stress field to the flow field of a compressible fluid is discussed, and the restrictions on the yield criteria are noted that stem from the necessity of a real velocity of sound in the fluid. It is shown that the analogy between shock waves and stress discontinuities indicated by Inoue is incomplete. The remainder of the paper is concerned with two-dimensional problems (plane stress or plane strain in the plastic solid). Inoue's basic equations are rederived along different lines and some particular correspondences are explored. In discussing possible applications of the analogy, the author emphasizes the difficulties created by the pressure of non-plastic zones and from the fact that the boundary conditions in plasticity are generally of mixed type, involving velocities and stresses. *W. Prager* (Providence, R. I.).

***Prager, William. On slow visco-plastic flow.** Studies in mathematics and mechanics presented to Richard von Mises, pp. 208–216. Academic Press Inc., New York, 1954. $9.00.

This paper considers an incompressible Bingham solid which may be defined as follows. If a Newtonian viscous fluid, a Mises perfectly plastic solid and a Bingham solid are each subjected to the same velocity strain, the stress in the Bingham solid is the sum of the stresses in the other two materials.

The general boundary-value problem is defined by giving the surface tractions over a part S_T of the boundary and the velocities over the remainder S_V. If neither S_T nor S_V are zero, the author proves there exists a unique velocity solution to the boundary-value problem and a unique stress solution in all nonrigid regions. If S_V is zero the solution is unique to within a rigid body motion, while if S_T is zero the solution is unique to within a constant hydrostatic pressure.

Two extremum principles are then proved. If a kinematically admissible velocity field is defined as any incompressible field which satisfies the boundary conditions on S_V (assumed not to vanish), then among all such fields the actual one minimizes a certain integral function of the strain invariants. Similarly, if a statically admissible stress field is any field which is in internal equilibrium and satisfies the

boundary conditions on S_T (assumed not to vanish) then among all such fields the actual one maximises a certain integral function of the stress invariants.

Both the Newtonian fluid and the Mises solid may be obtained as limiting cases of the Bingham solid. In this way previously obtained extremum principles are recovered, except in the case of the maximum principle applied to a Newtonian fluid where the result appears to be new. However, although the principles can be given physical interpretations in terms of energy dissipation in the limiting cases, no such simple interpretation appears to be available for the general Bingham solid. *P. G. Hodge, Jr.*

MATHEMATICAL PHYSICS

Have

★Courant, R., and Hilbert, D. Methods of mathematical physics. Vol. I. Interscience Publishers, Inc., New York, N. Y., 1953. xv+561 pp. $9.50.

This book is a welcome translation of the second edition of the well known "Methoden der mathematischen Physik" [Springer, Berlin, 1931]. The text covers the following subjects: linear transformations and quadratic forms, development of arbitrary functions in series of orthogonal functions, linear integral equations, calculus of variations, eigenvalue and vibration problems, application of variational calculus to eigenvalue problems, and special functions, as in the German original. The main additions are an interesting appendix by W. Magnus, treating the transformation of linearly independent spherical harmonics in three variables under a rotation of the coordinate system and a paragraph entitled "Reciprocal quadratic variational problems" (chapter 4, §11, pp. 252–257), which amplifies the discussion of upper and lower bounds of quadratic functionals and Friedrichs' analysis of Trefftz's method given in §9 (in this connection, a remark of J. B. Diaz and A. Weinstein [J. Math. Physics **26**, 133–136 (1947); these Rev. **9**, 211] seems to have been overlooked). *J. B. Diaz.*

Döring, W. Über die Zusammenhänge zwischen den verschiedenen physikalischen Begriffssystemen. Z. Physik **138**, 290–300 (1954).

The author expounds the distinction between physical concepts (Begriff), referring to any measurable entity, on the one hand and the actual measurement on the other. To clarify the discussion of the effect of a change in the set of basic concepts, he proposes to classify concepts according to "character", this being a finer classification than that of dimensionality (cf. e.g. energy and torque). These ideas are discussed in detail in regard to the use of five or fewer basic concepts in electrodynamics, with special reference to charge and capacity. The paper is one of a group dedicated to R. W. Pohl, and arises from ideas in Pohl's textbooks [see also R. Fleischmann, Z. Physik **129**, 377–400 (1951); these Rev. **13**, 199; and the paper reviewed below]. *F. V. Atkinson* (Ibadan).

Fleischmann, R. Das physikalische Begriffssystem als mehrdimensionales Punktgitter. Z. Physik **138**, 301–308 (1954).

The author sets up a pictorial representation of the dimensionality of physical concepts. If the entity $\mathbf{X}$ is representable dimensionally in the form $\prod_1^n \mathbf{B}_r^{\alpha_r}$, the α_r being integral and unique, then $\mathbf{X}$ corresponds to the point $(\alpha_1, \cdots, \alpha_n)$ in an n-dimensional lattice, and the $\mathbf{B}_r$ are termed a basis system. Basis systems correspond to elementary cells in the lattice. An interesting application is given to electrodynamics, where theories which use less than five basic quantities can be viewed as projections of the five-dimensional lattice upon one of lower dimension; the effect of such projections upon systems of units is also discussed. *F. V. Atkinson* (Ibadan).

Drobot, S. On dimensional analysis. Zastosowania Mat. **1**, 233–272 (1954). (Polish. Russian and English summaries)

This paper expands, for the nonmathematician, its earlier version in English [Studia Math. **14**, 84–99 (1953); these Rev. **16**, 96]. The dimensional quantities ("measures") A are combined with the positive real numbers α into a vector space Π, a linear combination of the vectors X and Y being X^aY^b (a, b arbitrary real), and the number 1 being the zero vector. The measures A_i are dimensionally independent if no linear combination of them is a number. Let Θ be a linear transformation in Π leaving every number invariant. A number- or measure-valued function Φ of the measures A_i is dimensionally invariant if, and only if, $\Theta\Phi(A_i)=\Phi(\Theta A_i)$. If the A_i are dimensionally independent, such a function is linear in Π. Φ is dimensionally homogeneous if

$$\Phi(\alpha_i A_i)=\alpha\Phi(A_i),$$

where α is determined by the α_i alone. It is then shown that, if Φ is dimensionally invariant and homogeneous, it is given by $\alpha A_1^{a_1}\cdots A_m^{a_m}$, where α is a function of nondimensional linear combinations of the A_i. This the author calls "the Buckingham Pi theorem," and argues informally that no stronger theorem has ever been proved. The reviewer believes that the assumption of dimensional invariance makes the theorem very weak, and that stronger versions of the theorem have been proved [cf. G. Birkhoff, Hydrodynamics, Princeton, 1950, p. 82; these Rev. **12**, 365] in which invariance is required only under a subgroup of the group $A\rightarrow\alpha A$, α real positive. The author does not draw a distinction between the invariance of a function and that of an equation, and his criticisms overlook the fact that a function of measures A_i becomes a function of numbers α_i by putting $A_i=\alpha_i E_i$, A_i and E_i having the same dimensions. The remainder of the paper consists of examples, a discussion of the well-known Riabouchinsky paradox, and some remarks about modeling. *A. W. Wundheiler* (Chicago, Ill.).

Eskenazi, Moiz. Exposé et critique de la *Théorie des Grandeurs* du professeur Max Landolt. Bull. Tech. Univ. Istanbul **5** (1952), 17–26 (1953). (Turkish summary)

M. Landolt's "Grandeur, mesure, et unité" [Paris, Dunod, 1947; translation of "Grösse, Masszahl und Einheit", Rascher, Zürich, 1943] is one of the recent attempts to set the theory of physical dimensions upon an axiomatic basis. The book itself is lengthened by many examples and much discussion; the author of the paper being reviewed devotes six pages to a formal exposition of Landolt's theory. There are two fundamental operations, one to represent combination of two magnitudes of the same dimension so as to form another of the same dimension, the other to denote generation of a quantity of new dimension. Under these two operations, dimensional quantities which are powers under the first operation of the unit element for the second operation are asserted to form a field isomorphic to

the real field. [In the paper under review, only integer powers are defined; in the work of Landolt, we find general real powers used without any definition, although on p. 65 we read "nous postulons la validité des règles des puissances lorsque les mesures sont des nombres réels," and on p. 69 a similar statement.] The author expresses two objections to Landolt's theory: (1) it does not adequately represent the physical situation desired, since it permits us always to replace two quantities of like dimension by another, and (2) Landolt without physical justification has extrapolated properties of the real numbers to the calculus of dimensions. To illustrate the first objection, he remarks that two weights on a single balance pan are indeed equivalent to a single one, but two weights on the two pans are not equivalent to a single one. He concludes that the problem of axiomatization of dimensional analysis remains open. [What seems to the reviewer a more satisfactory set of axioms has been given by S. Drobot, Studia Math. 14, 84–99 (1953); these Rev. 16, 96.] *C. Truesdell* (Bloomington, Ind.).

van Dantzig, D. On the geometrical representation of elementary physical objects and the relations between geometry and physics. Nieuw Arch. Wiskunde (3) 2, 73–89 (1954).

The author is interested in the question how far we can come in physics without making use of any metrical assumptions at all, restricting the geometrical tools used to those provided by affine geometry. Thus velocity and acceleration are contravariant vectors, but force (defined by invariant work) is covariant. A similar analysis is made of electromagnetic bivectors and basic equations are written out in a vector notation which uses arrowheads above and below for contravariant and covariant, and a tilde for density. The whole discussion is inspired by the work of J. A. Schouten, to whom the paper is dedicated. It ends with a discussion in very general terms of the relations between physics and geometry. *J. L. Synge* (Dublin).

Optics, Electromagnetic Theory

Kopal, Zdeněk. Photometric effects of reflection in close binary systems. Monthly Not. Roy. Astr. Soc. 114, 101–117 (1954).

The author considers the problem of the mutual illumination of two spheres under Lambert's law of reflection. The computations are carried out through the fourth power of the ratios of the radii to the center distance of the spheres, since this is the degree of accuracy to which the rotational and tidal distortion of the spherical shapes of the two bodies are negligible. The law of the variation of reflected light with the phase is derived as the principal result of the paper. *R. G. Langebartel* (Urbana, Ill.).

de Hoop, A. T. On the scalar diffraction by a circular aperture in an infinite plane screen. Appl. Sci. Research B. 4, 151–160 (1954).

Levine and Schwinger's variational formulation of the diffraction of monochromatic plane waves incident normally on a circular aperture in a perfectly soft plane screen leads to the problem of solving an infinite system of linear equations for the coefficients in a suitable expansion of the aperture function. Levine and Schwinger [Physical Rev. (2) 74, 958–974 (1948); these Rev. 10, 221] used an expansion of the form $\sum_{n=1}^{\infty} a_n(1-\rho^2/a^2)^{n-\frac{1}{2}}$, where a is the radius of the aperture, ρ distance from its centre. Bouwkamp suggested that a more appropriate expansion is $\sum_{n=0}^{\infty} b_n P_{2n+1}[(1-\rho^2/a^2)^{\frac{1}{2}}]$, where P_{2n+1} denotes the Legendre polynomial of odd order. In the present paper the system of equations in b_n is investigated. *E. T. Copson* (St. Andrews).

Braunbek, Werner. Zur Beugung an der kreisförmigen Öffnung. Z. Physik 138, 80–88 (1954).

In studying the diffraction of a scalar plane wave by a circular aperture in an infinite plane screen, for the case of normal incidence and wave function vanishing on the screen, the author [Z. Physik 127, 381–390, 405–415 (1950); these Rev. 12, 223] developed two different procedures of improving the modified Kirchhoff solution in the limiting case $ka\to\infty$, where a is the radius of the aperture and k the wave number, by estimating the contribution of the narrow zone along the edge of the aperture on the basis of Sommerfeld's solution for the half-plane problem. It was claimed that the two procedures give identical results along the axis of the aperture, except for terms of order $1/ka$. However, this turned out to be incorrect [C. J. Bouwkamp, Mathematics Research Group, Washington Square College of Arts and Science, New York University, Research Rep. No. **EM-50** (1953); cf. also Reports on Progress in Physics 17, 35–100 (1954), sect. 5; these Rev. 14, 1148; 16, 200]. In the paper under review the author carefully examines the cause of this discrepancy. He now shows that his claim is justified except in the immediate vicinity of the centre of the aperture. The exceptional region becomes smaller and smaller as ka tends to infinity. *C. J. Bouwkamp* (Eindhoven).

Papadopoulos, V. M. Propagation of electromagnetic waves in cylindrical wave-guides with imperfectly conducting walls. Quart. J. Mech. Appl. Math. 7, 326–334 (1954).

The author uses a perturbation method to calculate the effect of imperfectly conducting walls on the propagation of electromagnetic waves in cylindrical wave-guides of arbitrary cross-section. In cases where a particular type of degeneracy occurs in the ideal guide, the introduction of imperfectly conducting walls removes the degeneracy. *A. E. Heins* (Pittsburgh, Pa.).

Bremmer, H. The extension of Sommerfeld's formula for the propagation of radio waves over a flat earth, to different conductivities of the soil. Physica 20, 441–460 (1954).

The effect of the ground is introduced through an impedance-type boundary condition at the surface of the earth. The problem is thus reduced to the solution of the reduced wave equation in the half-space above the (x, y)-plane, with a point source and the boundary condition $\partial\pi/\partial z = -(2ik_0)^{1/2}\mu(x, y)\pi$. By the use of Green's theorem and saddle-point approximation, the problem is further reduced to the solution of the integral equation

$$\Psi(x) = x^{-1/2} + i\pi^{-1/2}\int_0^x \mu(\xi)(x-\xi)^{-1/2}\Psi(\xi)\, d\xi.$$

The parameter μ (depending on soil conditions) thus need not be known at every point (x, y) but only on the straight line joining the transmitter and receiver. This integral equation is then solved in the special cases of μ assuming different but constant values in two and three regions by

operational methods. In the case of two regions the solution is identical with that obtained by a different method by Clemmow [Philos. Trans. Roy. Soc. London. Ser. A. **246**, 1–55 (1953); these Rev. **14**, 1149]. *J. Shmoys.*

De Socio, Marialuisa. Sulle frequenze critiche in una guida d'onda. Boll. Un. Mat. Ital. (3) **9**, 283–285 (1954).

An application of the isoperimetric inequalities of Pólya and Szegö [Isoperimetric inequalities in mathematical physics, Princeton, 1951; these Rev. **13**, 270] enables the author to deduce some limitations on the critical frequency of guided electromagnetic waves. *A. E. Heins.*

Baudoux, Pierre. Guides d'ondes à conductivité finie. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 990–994 (1954).

Müller, Claus. Radiation patterns and radiation fields. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. **EM-62**, i+16 pp. (1954).

Let r, ϑ, ϕ denote spherical coordinates. Let U denote a solution of the scalar wave equation $\Delta U+k^2U=0$, regular for $r>c$ and obeying Sommerfeld's radiation conditions: $r(\partial u/\partial r-ikU)\rightarrow 0$, rU bounded as $r\rightarrow\infty$. The author shows: (i) U is expansible in a series of Legendre and spherical Bessel functions uniformly convergent if $r\geqq c'>c$; (ii) the leading term, $r^{-1}\exp(ikr)f(\vartheta,\phi)$, of the asymptotic expansion of U as $r\rightarrow\infty$ is correctly obtained from the series expansion if the spherical Hankel functions are replaced by their leading asymptotic terms. [This procedure may be justified in a simple manner by evaluating the integral of equation (10) of the paper asymptotically.] With the "radiation pattern" $f(\vartheta,\phi)$ the author associates an integral harmonic function $H(\mathbf{r})$ such that on the unit sphere H and f coincide and further that

$$\int\int_{|\varrho|=r}|H(\varrho)|^2dS=O\{e^{2k(c+\epsilon)r}\}\text{ for all }\epsilon>0\text{ as }r\rightarrow\infty.$$

Conversely, he shows that the latter relation is sufficient to ensure that $f(\vartheta,\phi)$ is the radiation pattern of some U. It is emphasised that not every continuous function f is a possible radiation pattern. On the other hand, any continuous f can be approximated to any degree of exactness by suitably chosen sources in an arbitrarily small sphere. [Misprints: a factor i is missing in the right-hand sides of equations (7) and (11); for $r>0$, in lemma 2, read $r>C$. Similar work by the same author has appeared in paper no. 18 in the prepublication of volume I of the Proceedings on microwave optics, June 1953, The Eaton Electronics Research Laboratory, McGill University, Montreal, Canada.]
C. J. Bouwkamp (Eindhoven).

***Knudsen, H. Lottrup. Bidrag til teorien for antennesystemer med hel eller delvis rotationssymmetri. [Contribution to the theory of antenna systems with complete or partial rotational symmetry.]** Teknisk Forlag, København, 1953. vi+228 pp. (32 plates). 35 kr.

Nisbet, A., and Wolf, E. On linearly polarized electromagnetic waves of arbitrary form. Proc. Cambridge Philos. Soc. **50**, 614–622 (1954).

For the case of time-harmonic electromagnetic fields whose electric or magnetic vector is linearly polarized everywhere, the authors derive differential equations relating the vector amplitude and phase of the field. The phase is found to satisfy a generalized eikonal equation, i.e., one with an additional term depending on frequency and amplitude. The amplitude vector satisfies a transport equation along an orthogonal trajectory of equiphase surfaces. The Poynting vector is also discussed. *J. Shmoys.*

Poincelot, Paul. Sur la répartition du courant le long d'un radiateur cylindrique. C. R. Acad. Sci. Paris **239**, 1365–1367 (1954).

Poincelot, Paul. Sur la répartition du courant le long d'une antenne cylindrique à l'emission. C. R. Acad. Sci. Paris **239**, 1472–1474 (1954).

Garabedian, P. R. An integral equation governing electromagnetic waves. Quart. Appl. Math. **12**, 428–433 (1955).

In electromagnetic theory. the problem arises of determining in the region D exterior to a simple closed curve C a complex-valued solution of $\partial^2u/\partial\xi^2+\partial^2u/\eta^2+u=0$ which takes prescribed values on C and satisfies the radiation condition $\lim_{\rho\rightarrow\infty}\rho^{1/2}(\partial u/\partial\rho-iu)=0$, where $\rho^2=\xi^2+\eta^2$. The object of this note is to prove the existence of such a solution.

The idea is to map D conformally on the region $|z|>R$ by a transformation $\zeta=z+f(z)$, where $z=x+iy$, $\zeta=\xi+i\eta$ and $f(z)$ is an analytic function regular at infinity. Then we have to prove the existence of a solution of

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}+|1+f'(z)|^2U=0$$

which takes prescribed values on $|z|=R$ and satisfies the condition $\lim_{r\rightarrow\infty}r^{1/2}(\partial U/\partial r-iU)=0$, where $r^2=x^2+y^2$.

If $G(z,w)$ is the Green's function for the region $|z|=R$ with singularity w associated with the differential equation $\partial^2U/\partial x^2+\partial^2U/\partial y^2+U=0$, then

$$\int_{|z|=R}U(z)\frac{\partial G(z,w)}{\partial r}ds=2\pi U(w)$$

$$-\int\int_{|z|>R}U(z)G(z,w)\{p(z)\}^2dxdy,$$

where $\{p(z)\}^2=|1+f'(z)|^2-1$. Since the left-hand side is a known function, this is a Fredholm integral equation for the unknown function $U(z)$ when $|z|>R$. This equation can be written as

$$g(w)=V(w)-\int\int_{|z|>R}V(z)K(z,w)dxdy,$$

where $g(w)$ is a known function and

$$V(z)=p(z)U(z),\quad 2\pi K(z,w)=p(z)p(w)G(z,w).$$

This equation has symmetric kernel of integrable square, to which the Fredholm theory can be applied, and the existence of a unique solution $V(z)$ of integrable square follows.
E. T. Copson (St. Andrews).

***Lopuhin, V. M. Vozbuždenie èlektromagnitnyh kolebanii i voln èlektronnymi potokami. [Excitation of electromagnetic oscillations and waves by electron currents.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 324 pp. 10.50 rubles.

According to the foreword, the main purpose of this book devoted to the study of the interaction of high frequency

electromagnetic fields and electron currents is to present the approximations which can be made in the formulation and solution of this problem. The author believes that this approach, involving as it does a precise statement of the physical conditions necessary for the validity of a given approximation, should enable the reader to appreciate and evaluate critically the literature of high-frequency electronics. To be covered in the book are the basic methods of generating and amplifying centimeter waves with special emphasis on the newer methods, such as travelling-wave tubes and electron wave tubes. Concrete physical approximations are to be illustrated by means of the analysis of a specific electron-beam device. With the emphasis placed on the physical meaning of the results, cumbersome mathematical derivations and transformations are to be avoided. The book is intended for students of radio-physics and for radio-engineers interested in high frequency electronics. To make the book suitable for a wide circle of readers in the above categories, the examples illustrating some particular method of calculation, as well as the basic terminology of high frequency electronics, are to be given in detail.

In view of the clear statement of intentions on the part of the author, the task of the reviewer is two-fold: 1. To determine how well the author has accomplished his purpose. 2. To determine the usefulness of the book as planned and as carried out.

1. The book seems to fulfill its purpose. The topics are treated lucidly, the various steps in the theory being made particularly clear. While the theoretical treatment starts from fundamentals, the text presupposes that the reader has a good practical background in electronics, i.e. is fully acquainted with the structure and operation of the tubes discussed, though not with the underlying theory. The mathematical level is that of advanced graduate students. The topics covered include: the excitation of cavity resonators, first in the approximation of a given field and then in the approximation of given currents, the general theory of single contour klystrons, the excitation of systems of recurring four-terminal sections, of systems with distributed parameters, and of an arbitrary resonator, the travelling wave tube, and the electron wave tube. The magnetron is mentioned only casually in the discussion of complex resonators. While a great deal of the basic theory of the magnetron is covered in that chapter, concrete physical examples are missing, which is unfortunate since the subject merits at least as much detail as the travelling wave tube. Another, more serious drawback is the absence of an index, which makes impractical the use of the book for occasional reference. The references to literature in Russian are plentiful, not so to publications in other languages.

2. The book should be valuable to anyone not having ready access to texts in English, French, or German or ready command of these languages. It may also be valuable to somebody interested in the development of the subject in the USSR. However, in view of such texts as Slater's "Microwave electronics" [Van Nostrand, New York, 1950], there is no need for a translation of the present book.

J. E. Rosenthal (Passaic, N. J.).

Gold, Louis. Relativistic dynamics of a charged particle in crossed magnetic and electric fields with application to the planar magnetron. J. Appl. Phys. **25**, 683–690 (1954).

In this and in the following paper the author has treated the problem of two-dimensional relativistic motion of an electron in constant electric and magnetic fields acting along the x and z axis, respectively. By eliminating the time from the equations a first-order non-linear equation is obtained between the reduced velocities β_y, β_x, which by integration yields a quadratic relation in β_y, β_x. The constant of integration C_0, entering in this equation, is determined by assuming zero initial velocity of the electrons, in which case C_0 depends only on the ratio of the fields E, H. With this assumption about the initial velocity of the electrons the author has analyzed the conditions for the existence of periodic orbits and their forms, and also the limiting case corresponding to aperiodic motion, from which he derives relations between transit time and the applied fields, the period of motion and the transversed distance. By introducing an auxiliary variable φ which depends linearly on β_y, the time t is easily expressed in terms of φ by a single quadrature. Furthermore, eliminating t from β_y and from the (φ, t) relation, the author expresses the coordinates x, y of the path of the electron in parametric form with respect to φ. *N. Chako* (New York, N. Y.).

Gold, Louis. On the nature of the transcendental curves associated with the relativistic trajectories of charged particles. J. Appl. Phys. **25**, 691–697 (1954).

The results of the paper reviewed above are extended to the case when the initial velocity of the electron (β_{0x}, β_{0y} at $t=0$) does not vanish. By means of a new auxiliary variable ϑ which reduces to φ of the cited paper, when $\beta_{0x}=\beta_{0y}=0$, the equation of the orbit of the electron is given in parametric form in ϑ. The time t is also expressed in ϑ. Both the position of the orbit (coordinates x, y) and t depend on the constant of integration C_0, which involves the initial velocity (β_{0x}, β_{0y}) and the ratio of the fields E, H. The limiting case (aperiodic motion) and the transit time and period of oscillation are expressed in terms of C_0 and the initial value of $\vartheta=\vartheta_0$ at $t=0$. Finally, the path distance of the electron is expressed by an integral with respect to ϑ_1 which is evaluated explicitly for the case when the initial velocity of the electron vanishes. [Many expressions and formulae in both papers contain errors; unfortunately they are too numerous to be included here.] *N. Chako* (New York, N. Y.).

Marziani, Marziano. Forze ponderomotrici nei dielettrici. Ann. Univ. Ferrara. Sez. VII. (N.S.) **1**, 47–54 (1951).

This paper deals with the calculation of ponderomotive forces in isotropic rigid dielectrics in an external electrostatic field. The purpose of the author is to show the equivalence between the force due to volume and surface density charges and the action of the field on the dipoles composing the dielectric body. The method described here is roughly this. By assuming the ponderomotive force F to be entirely due to the change of internal energy of the dielectric the author has derived an expression for F by calculating the difference in electrostatic energy of the initial and final configuration states of the dielectric body due to the external field. This is given by

$$F=\int_\tau \frac{dE}{dM}P\cdot\frac{\partial M}{\partial q}d\tau+\frac{1}{2}\int_\Sigma \frac{(P\cdot n)^2}{\epsilon_0}n\cdot\frac{\partial M}{\partial q}dS, \tag{1}$$

where E, P are the electric field and polarization, respectively, n is the normal to the surface of the dielectric, ϵ_0 is the dielectric constant of the surrounding medium and M, q are a point in the dielectric and parameters describing the configuration, respectively. Expression (1) has also been derived on the basis of the dipole model of the dielectric.

Finally, on the assumption that the dielectric is rigid, (1) can easily be transformed into

$$(2)\qquad F=-\int_\tau (\operatorname{div} P)E\cdot\frac{\partial M}{\partial q}d\tau+\int_\Sigma (P\cdot n)E_s\cdot\frac{\partial M}{\partial q}dS$$

with E_s as the value of the electric field on the surface Σ. Identifying $\operatorname{div} P\sim-\rho$ (volume charge density) and $(P\cdot n)\sim\sigma$ (surface charge density) the proof of the equivalence of the forces acting on volume and surface charge densities on one hand and the forces acting upon a polarized dielectric on the other has been established. *N. Chako.*

Marziani, Marziano. Sulle forze ponderomotrici nei dielettrici anisotropi. Rend. Sem. Mat. Univ. Padova **20**, 389–395 (1951).

In this article the author has extended the proof of the equivalence between the force acting on volume and surface density charges and the force acting on dipoles, which compose a dielectric body when placed in an external electrostatic field, to anisotropic rigid dielectrics. The treatment is similar to that of the paper reviewed above. *N. Chako.*

Taksar, I. M., and Plume, Z. Ya. Some boundary problems of the theory of impulsive magnetization. Akad. Nauk Latv. SSR. Trudy Inst. Fiz. **6**, 21–38 (1953). (Russian)

The authors treat the problem of longitudinal magnetization of an infinite, circularly cylindrical, conducting wire of unit radius by the application of an external magnetic field. The wire has constant conductivity and permeability within the radius a from the axis, and other, also constant, conductivity and permeability between a and the outer surface. Two cases of external field are considered: in one the field jumps from zero to unity at time $t=0$, while in the other the value unity is approached exponentially. The longitudinal magnetic field is said to satisfy the differential equation $\Delta H=4\pi\mu(r)\gamma(r)c^{-2}\partial H/\partial t$, H being continuous across discontinuities in μ and γ. The problem is solved by separation of variables. *J. Shmoys* (New York, N. Y.).

Rytov, S. M. The magnetic flux created by a dipole within a ferromagnetic circular wire. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **27**, 307–312 (1954). (Russian)

The author calculates the flux of the magnetic field through a circular loop concentric with a circularly cylindrical wire due to a magnetic dipole located within the wire, at some distance from the axis. The wire is assumed isotropic and to have an arbitrary permeability μ. The author shows how the flux can be obtained from the axially symmetric potential field u due to a ring source, u and $u(\partial u/\partial r)$ being continuous at the surface of the wire. An expression for the flux in terms of Bessel function integrals is thus obtained. This expression is then investigated in a number of special cases: μ very large, $\mu=1$, distance between loop and dipole large. *J. Shmoys* (New York, N. Y.).

Raisbeck, G. A definition of passive linear networks in terms of time and energy. J. Appl. Phys. **25**, 1510–1514 (1954).

According to the definition proposed by the author, a linear network is passive if (i) the total energy delivered to the network is not negative when currents of any waveform are applied at the network terminals and (ii) no voltages appear between any pair of terminals before a current is applied to the network. With this hypothesis it is shown that in the case of a two-terminal network the real part of the driving point impedance must be positive in the right half of the complex plane. No assumptions regarding rationality or reciprocity are made. This result is obtained by showing that with a particular excitation the real part of the impedance function is equal to the energy delivered to the network. In the case of an n-terminal network an analogous development shows that a certain Hermitian form must be positive definite. Thus for any network which satisfies (i) and (ii) the condition $\sum_{i,j} u_i^* u_j Z'_{ij}\geqq 0$ must hold, where the u's are arbitrary complex numbers, u_i^* is the conjugate of u_i and $Z'_{ij}=\frac{1}{2}(Z_{ij}+Z^*_{ij})$, and where Z_{ij} is an element of the network impedance matrix. Conversely, if the Hermitian quadratic form of a linear network is positive definite and no response appears before excitation is applied, then the total energy delivered to the network is not negative, implying the sufficiency condition.

R. Kahal (Monterey, Calif.).

Haus, H. A. Equivalent circuit for a passive nonreciprocal network. J. Appl. Phys. **25**, 1500–1502 (1954).

Lueg, Heinz. Die Mehrfach-Kurzschlussschieber-Messmethode zur Bestimmung der Transformationseigenschaften verlustloser $2n$-Pole zwischen homogenen Leitungen. Arch. Elektr. Übertragung **8**, 457–466 (1954).

A method for determining the input impedance at a pair of terminals of an n-port microwave structure when all other terminal pairs are terminated in arbitrary impedances is discussed. The terminating impedances are realized by pieces of short-circuited transmission lines and the input impedance is obtained from measurements on a line connected to the input terminals. The experimental procedure is described. *R. Kahal* (Monterey, Calif.).

Bader, Wilhelm. Rationale Gegenkopplungs- und Entzerrungsschaltungen oder Folgeregler mit vorgeschriebenen Eigenschaften. Arch. Elektr. Übertragung **8**, 285–296 (1954).

The problem considered is chiefly the synthesis of a feedback network to be used in conjunction with an amplifier. The properties of the complete system are prescribed and the gain function of the amplifier without feedback is presumed known. In the general case an auxiliary network may be required to alter the forward transmission properties. The significant results are stated in the form of three theorems which (i) give the restrictions on the overall system function necessary for a physical realization, (ii) treat the case when only the amplitude function is prescribed, and (iii) consider the servo problem separately. Methods for finding the feedback and auxiliary networks are discussed and examples given. *R. Kahal* (Monterey, Calif.).

Quantum Mechanics

***Blochinzew, D. I. Grundlagen der Quantenmechanik.** Deutscher Verlag der Wissenschaften, Berlin, 1953. xii+542 pp. DM 26.70.

This book is a German translation of a course of lectures given by the author at the Moscow Lomonossow University. It is a text officially approved by the USSR Ministry of Higher Education. Starting from very first principles, it introduces the reader to a fairly comprehensive understanding of non-relativistic quantum theory. The fundamental

experiments, both real and mental, are well described and the reader is not spared from useful mathematical tools such as linear operator theory, although it would seem premature to develop this in such detail before even mentioning the Schrödinger equation. As a result, simple problems such as the harmonic oscillator are not discussed until Chapter 8, which also includes a treatment of diatomic molecules and the effective mass of electrons in a periodic potential. Problems concerning spin, perturbation theory, the emission and absorption of light, many-body problems are all treated in great detail, and the technical part of the book concludes with a well-presented discussion of molecular forces and of ferromagnetism. In addition, there is a final section, "Some questions in the theory of knowledge", which attempts to point out the consistency of the theories of Schrödinger and Heisenberg with the theories of Lenin and Engels. Although this section was not too clear to the reviewer, it is independent of the rest of the book. *H. C. Corben.*

***Ahiezer, A. I., i Beresteckiĭ, V. B. Kvantovaya èlektrodinamika. [Quantum electrodynamics.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 428 pp. 18.30 rubles.

This is a comprehensive text-book covering the development of quantum electrodynamics from the beginning up to the end of 1952. It includes in particular a complete and very clear account of the new methods introduced by Tomonaga, Schwinger and Feynman during the years 1946–1950. It also contains a lot of useful material which has been previously published only in Russian journals, for example, in Chapter 1 the explicit construction due to L. Landau [Doklady Akad. Nauk SSSR (N.S.) **60**, 207–209 (1948)] of the wave-function of the two-photon system, and in Chapter 2 the discussion due to Beresteckiĭ [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 93–94 (1951)] of the relative parity of the electron and positron.

The chapter headings are as follows. 1. Quantum mechanics of the photon. 2. Relativistic quantum mechanics of the electron. 3. The quantized electromagnetic and electron-positron fields. 4. The general equations of quantum electrodynamics. 5. The scattering matrix. 6. The interaction of electrons with photons. 7. Retarded interaction of two charges. 8. Radiative corrections and vacuum polarization. Appendix 1. Theory of wave fields. Appendix 2. Equations for bound states. Appendix 3. Auxiliary mathematics. Roughly speaking, chapters 1–3 cover the theory of photons and electrons without interaction. Chapters 4–5 explain the general theory of interacting fields, the idea of renormalization of mass and charge, and the technique for eliminating divergences in the calculation of physical quantities. Chapters 6–8 contain applications of the general theory to the standard problems of radiation physics, including comparisons with experiment. Appendix 1 outlines the theory of relativistic wave-equations of particles of arbitrary spin, showing how these arise from the finite-dimensional representations of the Lorentz group.

The great strength of this book is the good balance which is maintained between physics and mathematics. The "highbrow theory" of chapters 4–5 is explained in adequate detail but does not push the more practical topics into the background. There is room in Chapter 7 for a thorough theoretical treatment of the internal conversion of gamma-rays of different electric and magnetic multipole orders. This is probably the most important phenomenon in quantum electrodynamics from the point of view of the experimentalist, and it is here discussed fully in a text-book for the first time.

At the places in the theory where mathematically rigorous argument is impossible (in the handling of divergent integrals) and where physical arguments are used to cover the gaps in the mathematics, the authors state clearly what they are doing. The difficulties are neither concealed nor over-emphasized. The point of view of the authors concerning these difficulties is the following. The theory gives a very accurate approximation to the behavior of a real electron, but it does not give a perfect or complete description of the electron since many important phenomena (e.g. mesons) are not taken into consideration. Since the theory is physically not perfect, it is not surprising to find also mathematical imperfections in it. The book ends with some quotations from Lenin which are unexpectedly relevant to the subject and support the moderate point of view of the authors.

There is unfortunately no index, but the lack is alleviated by a very full table of contents. In the reviewer's opinion, this is the first satisfactory text-book of modern quantum electrodynamics, and it is likely to remain for a long time the best. *F. J. Dyson* (Princeton, N. J.).

***Hund, Friedrich. Materie als Feld. Eine Einführung.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1954. viii+418 pp. Broschiert DM 48.00; ganzleinen DM 52.00.

This is an excellent introduction to quantum theory of matter fields. Starting from a review of the classical electromagnetic field, and elementary wave mechanics (chapters 1 and 2), the field picture of matter is rapidly built up. A thorough discussion of the concepts and methods of field theory is followed by a detailed treatment of non-interacting integral and ½-integral spin fields. The last two chapters briefly discuss the possible field interactions, and the relation of the theory to the presently known elementary particles.

The bulk of the work (7 out of 11 chapters) is devoted to introducing the concept of non-interacting matter fields. In this the book succeeds admirably. The treatment is didactic, distinguished throughout by a rare clarity. Even the most abstract notions, are introduced with a sense of purpose and concreteness. One very important feature is the connection with the aims and techniques of elementary wave mechanics which is never lost sight of.

The book is very weak when it considers interactions. The last two chapters, as far as their field-theoretical content is concerned, could have been written at any time before the advances of 1947 and after. This, however, does not detract from the very great value of the work, especially for the beginner who wants a first introduction to field theory. *A. Salam* (Cambridge, England).

Costa de Beauregard, Olivier. Théorie covariante relativiste de la solution de l'équation de Gordon. C. R. Acad. Sci. Paris **239**, 1357–1359 (1954).

Jouvet, Bernard. Equivalence de la théorie "complète" de Fermi avec la théorie de Yukawa. C. R. Acad. Sci. Paris **239**, 1267–1269 (1954).

Umezawa, Hiroomi, et Visconti, Antoine. Théorie générale des propagateurs. III. C. R. Acad. Sci. Paris **239**, 1466–1468 (1954).

Hugenholtz, N. M. Variational principle in quantum mechanics. Physical Rev. (2) **96**, 1158–1159 (1954).

Altshuler, Saul, and Carlson, J. F. Time-dependent variational principle. Physical Rev. (2) **95**, 546–548 (1954).

In the case of a time-dependent perturbation (as, for example, in the case of a collision in which the incident particle is regarded as a moving center of force) a variational principle is set up for the transition amplitude.

N. Rosen (Haifa).

Belinfante, Frederik J. Direct proof of the covariance of Gupta's indefinite metric in quantum electrodynamics. Physical Rev. (2) **96**, 780–787 (1954).

The metric of Gupta's formulation of quantum-electrodynamics is defined in terms of the odd or even character of the number of incident scalar photons. Since this number is not invariant, the covariance of the theory is not obvious. Here the covariance is proved in a direction fashion by explicitly computing the norm of a state for one observer in terms of the norms postulated by another.

H. C. Corben (Pittsburgh, Pa.).

Jordan, H. L., und Frahn, W. E. Nichtlokale Feldtheorie auf der Grundlage der Salpeter-Bethe-Gleichung. II. Wechselwirkung mit lokalisiertem Teilchen. Z. Naturforschung **9a**, 572–578 (1954).

In this continuation of part I [same Z. **8a**, 620–628 (1953); these Rev. **16**, 319] the authors investigate the interaction of a particle with another, localized, heavy particle. It is found that the interaction can be described by a local potential only in the lowest order of the perturbation calculation; the higher order terms lead to non-local potentials. Some examples of interaction terms are calculated and their behavior investigated. *N. Rosen* (Haifa).

Nicholson, A. F. On a theory due to I. Fényes. Australian J. Physics **7**, 14–21 (1954).

A criticism on physical grounds of work of Fényes [Z. Physik **132**, 81–106 (1952); these Rev. **15**, 78].

I. E. Segal (Chicago, Ill.).

Tani, Smio. Analysis of the structure of transformation function in quantum mechanics. Progress Theoret. Physics **11**, 190–206 (1954).

If the hamiltonian H is of the form $\sum h_n(t)M_n$, where the h are functions of time and the M are operators independent of t, then the Schroedinger equation may be solved by the ansatz $\psi=\exp[iG(t)]\chi$, with $G(t)$ expressed as an element of the Lie algebra, over functions of t, formed from M_n by taking the commutator as the Lie product. In problems for which explicit solutions can be found in the literature the above Lie algebra is of small finite dimension. This dimension is evidently a measure of the difficulty of the problem. A differential equation for G is obtained for the above exponential ansatz and also when the unitary transformation is given as a Cayley or half arctangent transform.

A. J. Coleman (Toronto, Ont.).

Imamura, Tsutomu, Sunakawa, Sigenobu, and Utiyama, Ryôyû. On the construction of S-matrix in Lagrangian formalism. Progress Theoret. Physics **11**, 291–308 (1954).

A non-local version of Schwinger's Lagrangian formalism [Physical Rev. (2) **82**, 914–927 (1951); these Rev. **13**, 520] is used to discuss the existence of an S-matrix for Yukawa's non-local theory. Conclusion: "There must be a new severe condition on the form factor if the S-matrix is to exist in the non-local case". *A. J. Coleman* (Toronto, Ont.).

Coester, F., Hamermesh, Morton, and Tanaka, Katsumi. Limiting processes in the formal theory of scattering. Physical Rev. (2) **96**, 1142–1143 (1954).

The unitary operator $U(t, t_0)$ occurring in scattering theory may be defined in terms of the adiabatic switching on of the interaction, with $t_0\to-\infty$, or in terms of an integral as proposed by Gell-Mann and Goldberger [Physical Rev. (2) **91**, 398–408 (1953); these Rev. **15**, 382]. In this paper these two definitions are shown to be equivalent.

H. C. Corben (Pittsburgh, Pa.).

Šapiro, I. S. On the parity of wave functions of para and ortho states. Doklady Akad. Nauk SSSR (N.S.) **95**, 975–977 (1954). (Russian)

It is known that the wave functions of para- and ortho-positronium, on the basis of their transformation properties, are a pseudoscalar and a vector, respectively [see, e.g., Beresteckiĭ, Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 93–94 (1951)]. The present note shows that these transformation properties hold for the S-states of any two spinor particles the wave functions of which transform in the same way under rotations and inversions and for which the square of the inversion matrix is equal to -1.

N. Rosen (Haifa).

Senitzky, I. R. Harmonic oscillator wave functions. Physical Rev. (2) **95**, 1115–1116 (1954).

The Schrödinger wave equation for the one-dimensional harmonic oscillator has been found to have a solution in the form of a wave packet that oscillates sinusoidally in time without any change of shape [E. Schrödinger, Naturwissenschaften **14**, 664–666 (1926)]. It is shown that there exists an infinite number of such solutions, each associated with a solution for a stationary state of the oscillator. The expectation value of the energy is found to be the sum of the quantum-mechanical energy of the stationary state plus the classical energy of the oscillating packet. *N. Rosen.*

Yang, C. N., and Mills, R. L. Conservation of isotopic spin and isotopic gauge invariance. Physical Rev. (2) **96**, 191–195 (1954).

It is pointed out that the usual principle of invariance under isotopic spin rotation is not consistent with the concept of localized fields. The possibility is explored of having invariance under local isotopic spin rotations. This leads to formulating a principle of isotopic gauge invariance, and the existence of a b field which has the same relation to the isotopic spin that the electromagnetic field has to electric charge. The b field satisfies non-linear differential equations. The quanta of the b field are particles of spin unity, isotopic spin unity and electric charge $\pm e$ or zero. (From the author's abstract.) *A. Salam* (Cambridge, England).

PART II IS AN INDEX WHICH HAS BEEN PHOTOGRAPHED AT THE BEGINNING OF THE VOLUME(S).